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SPECTRAL ENERGY CHARACTERISTICS OF THE MERCURY VAPOR LAMP

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ABSTRACT

By means of a large aperture quartz spectroradiometer the spectral energy distribution of a special variable-length mercury lamp was measured between 14,000 A and 2300 A under widely varying conditions. The effects of current, voltage and ventilation on the radiation at fifteen important maxima were measured, including three maxima in the infrared, four in the visible, and eight in the ultraviolet. The composition of each maximum is given, and curves are plotted showing the degree of resolution used; the variation in intensity of each maximum with voltage gradient at constant current; the variation of intensity of the strong ultraviolet maximum near 3660 A with voltage for five current values from two to four amperes; the stationary characteristic curves of a high pressure arc; two typical cases of spectral energy distribution at constant power; and the variation with current of the intensity of each maximum at constant power.

It was found that for constant current the energy in each maximum except those in the infrared increased linearly with the voltage after a certain minimum (near 8 volts for 2.5 amperes) was reached. The pressure was measured for each stationary state used, the extreme values being 20 mm of mercury and two atmospheres. Where power input was constant the energy radiated in each maximum increased rapidly with decreasing current. It was found that the most satisfactory condition for running an arc at pressures under two atmospheres, at least, was to have the pressure as high as possible, so that for a given power input the voltage was high and the current low. This condition gives the greatest efficiency and the least change in energy distribution with total energy variation. The relation of pressure to the other variables and the effects at higher pressures are being investigated further; also an absolute determination is being made of the energy available in each wave-length for photochemical purposes.

Although a number of papers on the quartz mercury vapor lamp have appeared since it was first designed some twenty years ago, surprisingly little information is available concerning the variation in its spectral energy distribution to be expected with changing conditions. Many workers have measured the relative intensities of the

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various lines in the visible and ultraviolet, but usually for only one or two power rates in a given lamp, and for conditions occurring only in their particular researches. It is, of course, not to be expected that a spectral energy curve for one arc will apply exactly to another, but sufficient data can be secured to determine, for example, what conditions should be chosen so that any variation of total energy radiated would produce the least variation in energy distribution.

Desiring a steady source of visible and ultraviolet light of intensity suitable for photochemical and photographic investigations, one whose energy distribution could be exactly determined in absolute units, we have studied the behavior of the quartz mercury lamp under widely varying conditions of use. While isolation of certain spectral regions by filters is sufficient for most commercial uses, the separation obtained in this way is not great enough for the photochemist. The present paper deals with the energy distribution obtained with a high aperture quartz prism spectroradiometer, and although the length of the spectrum from 11000 to 2100A was only three inches, sufficient resolution was obtained for the purpose desired, with small sacrifice of intensity.

The most complete investigation of the energy distribution of radiation from the quartz mercury lamp was made by Küch and Retschinsky[1] in 1906. They measured the total intensity of the visible, and the intensities of the individual visible lines, by means of a flicker photometer, and of the total ultraviolet radiation with a photoelectric cell, while a photographic method was used for the continuous background. They varied the power input into the lamp over a wide range, but made no effort to separate the variables, current and voltage. In other papers they give results of experiments determining the temperatures in various parts of the arc,[2] and the absorption by the luminescent mercury vapor of its own light.[3] Pflüger[4] has repeated and verified portions of their work, using a high aperture quartz-rocksalt spectrometer and a Rubens thermopile. He resolved the lines sufficiently to show that members of the same series increased in intensity together, which Küch and Retschinsky had predicted from their results. Athanasiu[5] also has recently verified this.

[1] Küch and Retschinsky, Ann. der Phys., *20*, p. 563; 1906.

[2] Ibid, *22*, p. 595; 1907.

[3] Ibid, *22*, p. 852; 1907.

[4] Pflüger, Ann. der Phys., *26*, p. 789; 1908.

[5] Athanasiu, Compt. Rend., *178*, p. 2071; 1924.

Ladenburg,[6] Fabry and Buisson,[7] Winther,[8] Souder,[9] Coblentz, Long and Kahler,[10] Coblentz and Kahler,[11] and Koppius[12] have measured the energy distributions in certain lamps for one or two power inputs.

Buttolph[13,14] has summed up the most important known characteristics of the mercury arc, traced the development of the quartz lamp, and discussed its theory and applications. He has collected a bibliography of papers dealing with the construction, characteristics, and uses of mercury arcs.

It is evident from a study of the results of these workers that a statement of the power put into a mercury lamp is not sufficient to determine the energy distribution of the radiation emitted by it. As with time and intensity in the photographic plate, a "reciprocity law" does not hold between current and voltage in the arc, since not only their product is important, but their ratio as well. In no published work, so far as we have been able to find, have the effects of current and voltage been separated definitely, by measuring the variation in energy distribution at constant current and variable voltage, and at variable current and constant voltage.

The purpose of the present work was to study the energy distribution from a typical lamp under widely varying conditions, to separate the effects of current and voltage, and to find what conditions, at high efficiency and high total intensity, were such that a small variation in total intensity would produce the least variation in energy distribution.

To get a clear picture of the relation of the variables concerned in the operation of the lamp, some discussion of its stationary characteristic curves is necessary. Fig. 1 shows several such characteristic curves marked T_1, T_2, and T_3. The stationary characteristic curve shows the current and voltage relations in the arc *when a steady state has been reached*. That is, for a given ventilation of the arc we get a curve T_1 when the line voltage is kept constant and various resistances R_1, R_2, etc. are placed in series with it. If, however, the arc is allowed freer

[6] Ladenburg, Phys. Zeits., *5*, p. 524; 1904.
[7] Fabry and Buisson, Compt. Rend., *153*, p. 93; 1911.
[8] Winther, Z. Electrochem., *20*, p. 109; 1914.
[9] Souder, Phys. Rev., *8*, p. 683; 1916.
[10] Coblentz, Long and Kahler, Bur. Stds. Sci. Paper No. 330; 1918.
[11] Coblentz and Kahler, Bur. Stds. Sci. Papers, 378, 233; 1920.
[12] Koppius, Phys. Rev., *18*, p. 443; 1921.
[13] Buttolph, Cooper-Hewitt Bull., 105.
[14] Buttolph, Gen. Elec. Rev., *23*, 741 and 858 and 909; 1920.

ventilation so that its heat can escape more rapidly, it will move along the stationary characteristic curve T_2 when its series resistance is changed. The steady state of the lamp is attained in a period varying from three minutes to an hour or more, according to conditions.

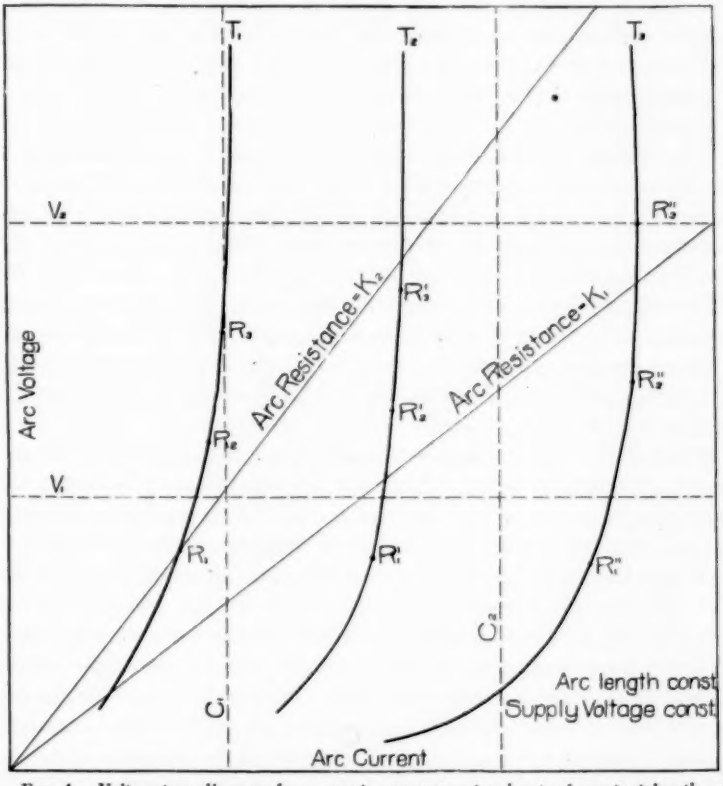


FIG. 1. *Volt-ampere diagram for a quartz mercury vapor lamp of constant length, with constant supply voltage.*

Curves T_1. . . T_3 are stationary characteristic curves, applying only after the lamp has come to a steady state. To pass from one of these to another a change must be made in the ventilation of any lamp.

It is evident that if we wish to study the lamp at constant current while varying its voltage, along line C_1, for instance, we must vary the ventilation of the lamp to do so. It is also evident that it is much more difficult to keep the voltage constant while varying the current, as along line V_1, since no part of any characteristic curve is so nearly

parallel to a constant voltage line, as to a constant current line. Under certain conditions the characteristic curves bend back, so that an increase in applied voltage produces a decrease in current; the lamp, when it gets too hot, goes out.

The usual type of mercury arc is built with constant length, and the curves in Fig. 1 assume a constant arc length. The arc used in the present work was designed to have variable length, for several reasons; all results are reduced to unit length however, after a correction of 12 volts had been made for the electrode drop.

The arc used is shown in Fig. 2. The legs of the inverted quartz U dip into two pyrex reservoirs, and are sealed to them with de Khotinsky cement, the upper part of each reservoir serving as a water cooler and the lower part containing the mercury for the arc. Iron wire electrodes are sealed into the sides of the reservoirs, and dip almost to the bottom of the mercury pools. A quartz trap is sealed to the top of the arc, so that the last trace of air in the tube can be exhausted by boiling the mercury up into the trap.

In the original model of the lamp the lower legs and the trap tube were of large bore, but the arc showed a great tendency to oscillate, the levels of the mercury moving up and down several centimeters continuously, while the arc voltage varied twenty per cent or more. In the final form these legs were made of quartz capillary of one millimeter internal diameter below the water coolers, which damped the oscillations down to a few hundredths of a millimeter and kept the voltage quite constant. A finer thread of mercury would not have carried the maximum current of five amperes.

The bends in the upper part of the quartz tube were made with thickened walls so that under the greater heat caused by deflected ions the tube would not fuse. The two reservoirs and the trap were connected through rubber pressure tubing by a system of stopcocks to vacuum, so arranged that any one of them could be separately exhausted either through an inch of thermometer capillary or through a tube of large bore. In this way exact control of all mercury levels was possible. Arrangements were also made to apply air pressure to the reservoirs, so that pressures up to two atmospheres could be reached. The pressures in the reservoirs could be read at any time from manometers connected with them.

In starting the arc, both reservoirs and the body of the lamp were exhausted to a fraction of a millimeter, and then the mercury was allowed to rise slowly in the arms of the tube. The outside of the

lamp, wherever it could be reached, was heated with a Bunsen flame and the mercury boiled vigorously. When the mercury had risen so as to partly fill the trap the rubber tube connecting this to the pump was closed with a pinchcock as near the trap as possible. The current

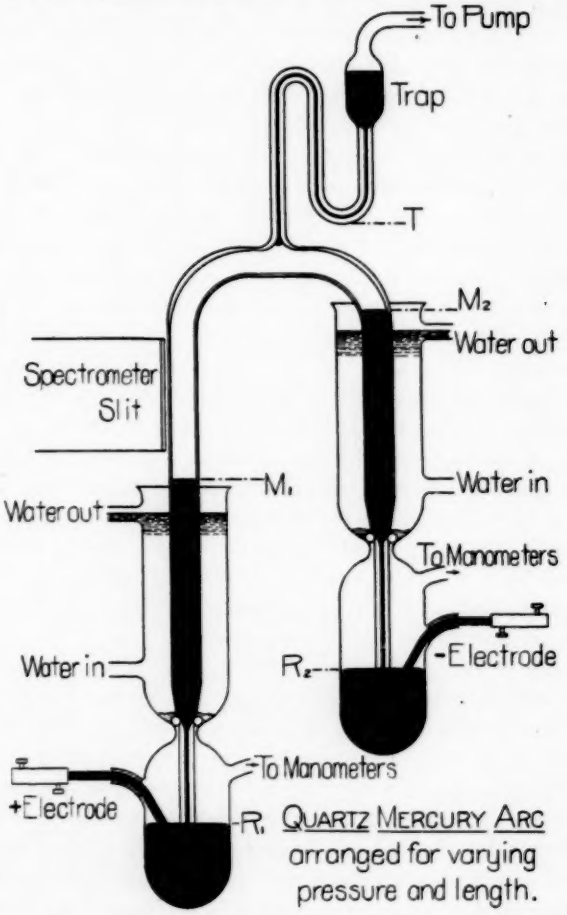


FIG. 2. *The special design of quartz mercury vapor lamp used in this work, arranged for variation of pressure, arc length, voltage gradient, current, series resistance, and rate of cooling.*

was then turned on, the upper electrode being always negative. Thus, the greatest flickering occurred at the surface farthest from the spectrometer slit. The two reservoirs were then slowly exhausted until the

mercury broke at the top of the U, and the arc length then increased as pressure was built up inside of the lamp. The height of the mercury columns was finally adjusted by varying the resistance in series with the lamp, the rate of cooling, and the pressure in the two reservoirs. Although the upper reservoir gradually filled at the expense of the lower, the excess could be returned without extinguishing the arc. A copper tube provided with small holes was bent to the shape of the quartz tube and currents of air from this were directed against the lamp over the whole luminous column, when necessary.

A lamp of this type possesses several important advantages. When the quartz becomes devitrified, after say 1500 hours of use, a short cylinder can be removed where the light is taken from the arc, and a fresh one put in its place. The lamp thus has an indefinite life. Then the arc length is variable, which is a desirable feature for certain work. The most important advantage of this type of arc, however, is that a change can be made from one stationary state to another on a different characteristic curve much more rapidly than in the ordinary arc. It was found possible to build up the pressure in the arc very rapidly, when first struck, by lengthening it to 25 cm, putting a large current through it, and then shortening the column thus building up the voltage drop. Where some twenty runs through the whole spectrum were taken in a day under various conditions it was very important to be able to arrive at the desired new state quickly.

The whole arc was surrounded by a large box with holes which could be opened or shut to provide different degrees of ventilation, the top of the box being open to a large pipe leading to a hood. A window of heavy glass was placed in one side of the box so that the arc could be observed from the outside, and a cathetometer was arranged to measure, while the arc was burning, the heights of all the mercury columns involved. All controls were outside of the box, so that it was possible to make runs continuously for a whole day, under widely varying conditions, without its removal.

In series with the arc were placed a fairly high inductance to damp out oscillations, and a resistance which would carry five amperes, variable from 200 ohms downward in small steps. The arc was fed by a 550 volt D.C. generator run on an induction motor; the terminal voltage was extremely constant. In no case, in the present work, was the voltage drop across the arc greater than 300 volts, which corresponded to about 17 volts per cm gradient. A precision ammeter was connected in series with the lamp, and a voltmeter across its terminals.

The spectroradiometer is shown in Fig. 3. The lamp, having a tube nine millimeters in internal diameter, was placed practically in contact with the slit, 4 cm long and variable in width from 1 mm to 0.05 mm. The slit was built upon the base of the instrument as a separate unit from the collimating lens, so that ample room for a shutter and several absorption cells was provided between them. In order to eliminate as much as possible the effect of stray infrared radiation the shutter was made of "electric smoke" glass, kindly loaned us by Mr. D. C. Stockbarger of the Massachusetts Institute of Technology, whom we wish to

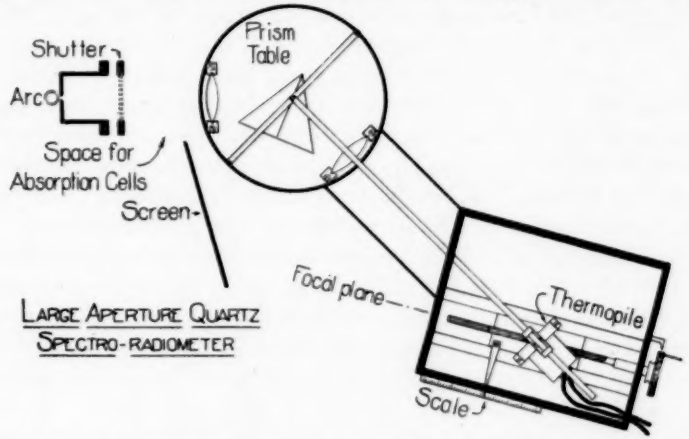


FIG. 3. *The spectroradiometer used to measure the intensity distribution in the spectrum of the quartz mercury vapor lamp.*

thank also for the loan of a resistance for the arc and for technical information regarding the Cooper-Hewitt mercury arc. This glass transmits the short-wave infrared, and is perfectly opaque to the visible and the ultraviolet, so that when null readings were taken in these regions, the shorter infrared penetrated the shutter, and thus cancelled out. Very little stray radiation was found, however.

The Cornu prism, made by Hilger, had a face nine cm long and six cm high. The two quartz lenses were six cm in diameter and twenty cm in focal length for D-light. The apparatus was collimated for about 3500A, and the prism was set at minimum deviation for this wavelength. The focal plane had a very small curvature.

The screw carrying the thermopile mounting was placed parallel to the focal plane, and with a motion of four inches the spectrum could be covered from 14,000A to 2000A. The thermopile mounting was free to

rotate about a vertical axis on its carriage, and a rod, passing through two guides in the top of the mounting and swivelled at its far end over the optical center of the prism, kept the hot junctions of the thermopile normal to the incident light. The prism and lenses remained fixed.

The thermopile was constructed of bismuth and silver, following the directions of Coblentz with certain minor variations. It contained fourteen junctions, and was used with a Leeds and Northrup high-sensitivity d'Arsonval galvanometer of 12 ohms resistance, 6 seconds period, and sensitivity 14.1 mm per microvolt. The junctions were 1 mm wide, but in most cases were covered with a slit .4 mm wide, and the deflections for single maxima were in many cases greater than 180 cm, which was the limit of the scale. This was 3.5 meters from the galvanometer, and was read with a telescope, a one-fourth inch mirror being used. The effective length of the thermopile slit was 18 mm, so that the entire length of the collimating slit was not used. The thermopile-galvanometer combination was calibrated in terms of a standard lamp from the Bureau of Standards, so that all measurements could be reduced to absolute energy values if desired. The present report deals only with relative values, however.

In Fig. 4 are shown two typical runs of the thermopile through the spectrum. Although the screw could be turned by hand, and the radiometer slit set on any desired wave-length, it was also arranged to be turned by a small synchronous motor properly geared down. The whole spectrum could be covered in any desired time from two minutes to fourteen, but in the present work the time was uniformly eight minutes, giving the highest speed consistent with negligible galvanometer lag. One observer sat at the galvanometer telescope and recorded the readings at all maxima, or at regular intervals, occasionally closing the shutter and taking a null reading; he also read the voltmeter just after each maximum was passed. The other observer measured the pressures registered for both sides of the arc. He also read the ammeter at stated intervals, and the heights of the arc mercury columns with the cathetometer. The bottom of the U in the trap, marked T in Fig. 2, was taken as a bench mark, and the other heights M_1 and M_2, bounding the arc, and R_1 and R_2, were then measured from this. From these readings, those of the external manometers, and the barometric height the arc pressure was computed on each side, and the results averaged. A full-size chart of the lamp was drawn to scale, and the arc length was taken from a table computed from this. This length

was measured at the center of the tube, where the thread of light is concentrated when the pressure is high.

Fig. 4 shows the degree of resolution used, which, although not high, is sufficient for most photochemical purposes, and is much greater than any obtainable with the usual filters. Greater resolution would have

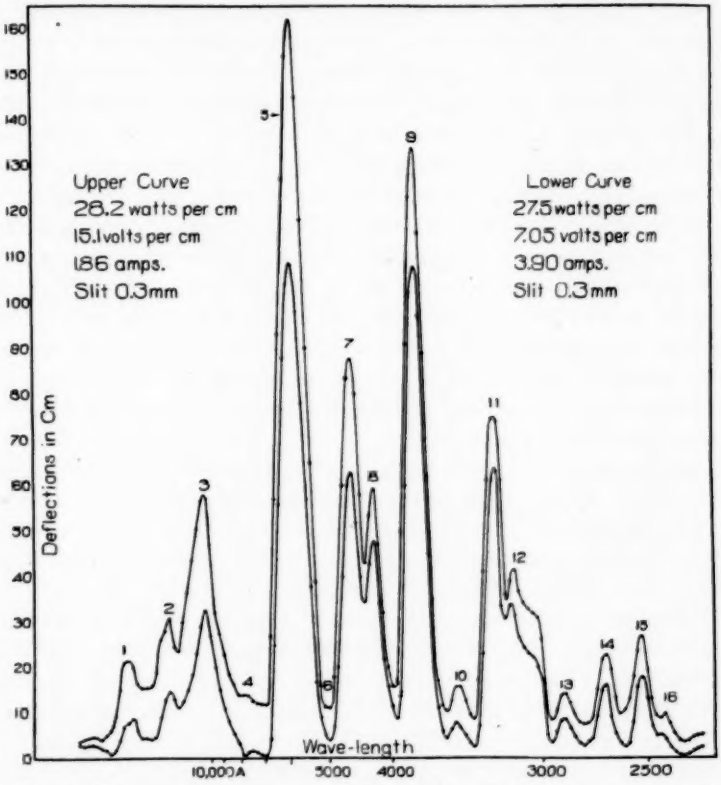


FIG. 4 *Two energy distribution curves taken for approximately the same power inputs, showing the maxima measured, the degree of resolution obtained, and the increase in intensity obtainable by using high voltages and small currents for a given power.*

been obtainable by sacrificing light intensity. Usually only the maximum and null values were read, but the curves of Fig. 4 were taken with readings made every four seconds.

It should be noted that the curves are for practically identical power inputs into the lamp, the difference being that the upper was for low current, high voltage and pressure, and the lower for high current, low voltage and pressure.

Table 1 gives the approximate values of the principal lines making up each maximum. The first column contains the arbitrary number

TABLE 1

Maximum	Lines	Series (Fowler)	Approximate Rel. Intensity
1, 2	Selective quartz radiation?		
3	General radiation		
	11288 A	$1s-2p_1$	2
	10140	$1P-2S$	5
4	6234	$2S-4P$	1
	6152	—	2
5	5819	—	3
	5804	$1s-4p_1$	3
	5790	$1P-2D$	3
	5769	$1P-2d'$	6
	5461	$1p_1-1s$	5
7	4359	$1p_2-1s$	5
	4348	$1P-3D$	5
8	4078	$1p_2-2S$	3
	4047	$1p_2-1s$	3
9	3663	$1p_1-2d''$	4
	3655	$1p_1-2d'$	6
	3650	$1p_1-2d$	10
10	3352	—	
	3342	$1p_1-2s$	
11	3132	$1p_2-2d''$	4
	3126	$1p_2-2d'$	5
12	3026	$1p_1-3d''$	1
	3024	$1p_1-3d'$	2
	3022	$1p_1-3d$	2
13	2804 etc.	$1p_1-4d$	1
		$-4d'$	0
		$-4d''$	0
14	2700	—	2
15	2537	$1S-1p_2$	2
16	2483 etc.	$1p_2-4d''$ etc.	1

given to the maximum, the second the wave-lengths of the lines included in it, the third their series notation according to Fowler's Report on Series in Line Spectra, the last a rough estimate of the relative intensities of the various members of each group. Since the heights of the various maxima were measured, and not their areas, the effects observed may be considered as being due to the strongest line in each case, with some effect due to the weaker lines. Since, however, the large ultraviolet maximum, No. 9, near 3660, is composed of lines belonging to one series only, members of a single diffuse triplet, all of these should increase together, and it may be considered as a single line. Its behavior is indistinguishable from that of the other maxima, those in the infrared excepted.

In comparing the relative intensities of the va ious maxima it should be remembered that no correction has been made for reflection or absorption in the optical train, or for the effective width of the slit which is greater at longer wave-lengths than at shorter. This correction would have to be made in studying the true spectral energy distribution from the lamp; we are here interested only in measuring the varying intensities of single lines.

In Fig. 5 are given curves showing how the various maxima increase in intensity as the voltage gradient in the arc is increased at constant current. In order to keep the current constant the degree of cooling of the lamp had to be varied in most cases. But under ordinary ventilation, with no forced draft, the lamp tended to draw about 2.5 amperes over a large voltage range. It is evident that the intensity of any maximum goes up linearly with the voltage gradient after a particular limiting voltage, varying with the individual lines, and of course with the current, has been reached. It seems possible that the voltage-energy curves may bend over again after a certain pressure has been reached, and this point is now being investigated with a very high pressure arc. Maximum 3 (10, 140A) is the only one which shows a slope not proportional to its relative height, and the increase is evidently due to general radiation from the hot quartz of the lamp. Table 2 gives the ratios of the values of the maxima at various voltages for constant current of 2.5 amperes. The slopes of the straight portions of the curves are also given, and under the heading "intercept" is listed the voltage gradient at which the straight portion of each curve would cut the voltage axis if it continued at the same slope. It is evident that the changing spectral energy distribution at low voltages is due largely to changing slope of the energy-voltage curves. At high voltage, how-

ever, the change is determined mainly by the different intercepts which the straight portions of the curve would have if extrapolated to the voltage axis. These intercepts are grouped within the region 4-7 volts, and undoubtedly have an important physical meaning, which should throw light on the mechanism of the mercury arc if interpreted. Before

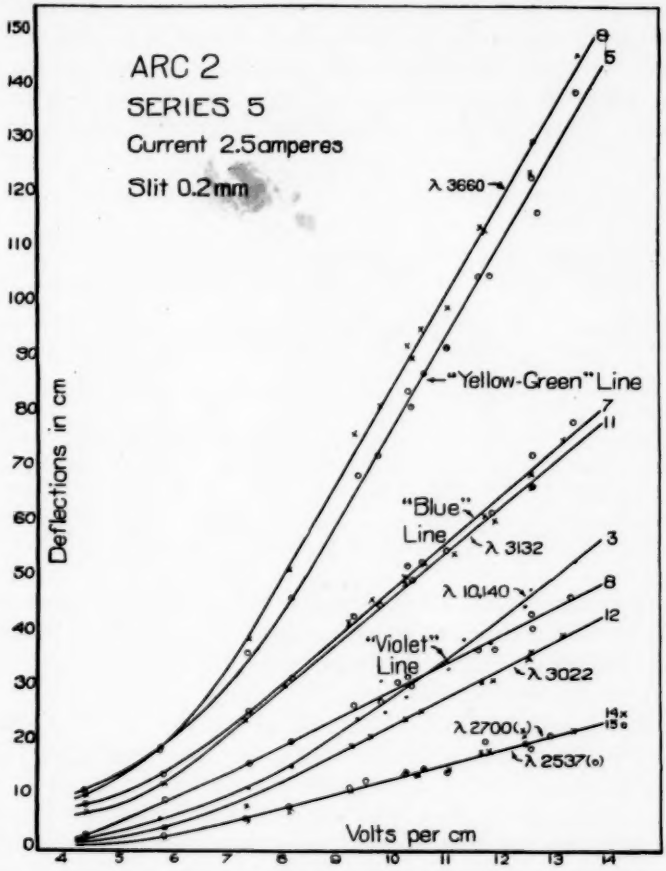


FIG. 5. *Curves showing the variation of energy with voltage gradient at constant current for ten of the most important maxima.*

definite interpretation can be made, however, the maxima must be separated into their individual lines.

Fig. 5 shows how rapidly intensity increases with voltage. Taking the strong ultraviolet line near 3660, for example, by increasing the

voltage three and one-half fold, from 4 v/cm to 14 v/cm, the energy of the line is increased from 9 to 154 units, or 17.1 times. Between 7 and 14 volts, where the energy-voltage curve is linear a 4.5-fold increment is produced due to the fact that the intercept is positive.

The degree of variation of individual points from the plotted curve must not be taken as an indication of the accuracy attainable. The variation is due to the fact that those points lying farthest from the curve correspond to states in the arc which had not become quite stationary. It was possible to test such points by plotting pressure against voltage for them; it was found in general that the greatest

TABLE 2

Maximum	Slope	Intercept volts	Ratio to Maximum 5 (V-G line) at				
			5 v	7 v	9 v	11 v	14 v
3	37°	6.4	.347	.337	.359	.375	.402
5	59°	5.6	1.000	–	–	–	–
7	40°	4.5	.725	.738	.655	.600	.562
8	27°	4.25	.375	.474	.407	.369	.345
9	59°	5.15	.909	1.100	1.120	1.080	1.050
10	40°	4.45	.587	.705	.638	.545	.545
12	27°	5.5	.173	.279	.303	.332	.329
14	15°	5.0	.013	.147	.168	.166	.165
15	16°	5.1	.013	.170	.180	.171	.165

errors occurred when a large pressure change had just taken place. Without the variable length feature, however, it would have been much more difficult to produce the desired change of stationary state in the short time allowed between runs.

Fig. 6 gives a family of energy-voltage curves for different current strengths for maximum 9, the strong ultraviolet triplet near 3660. This was chosen because of its simple structure and its intensity. Data for curves similar to these were taken for all maxima, but this set is representative. Voltage-energy curves were taken for each current strength separately, and current-energy curves at constant voltage were then made, these being used to tie the former set of curves together. The points taken from the double network of readings were checked together, and the final curves were plotted from a graphical average. The dotted curves in Fig. 6 show the variation in radiation of maximum 9 for two power inputs, with variable voltage. If a voltage-current

"reciprocity law" held these dotted curves would be horizontal and straight.

Fig. 7 shows the variation in energy radiated in each maximum when the power input per cm is kept constant, but the current and voltage

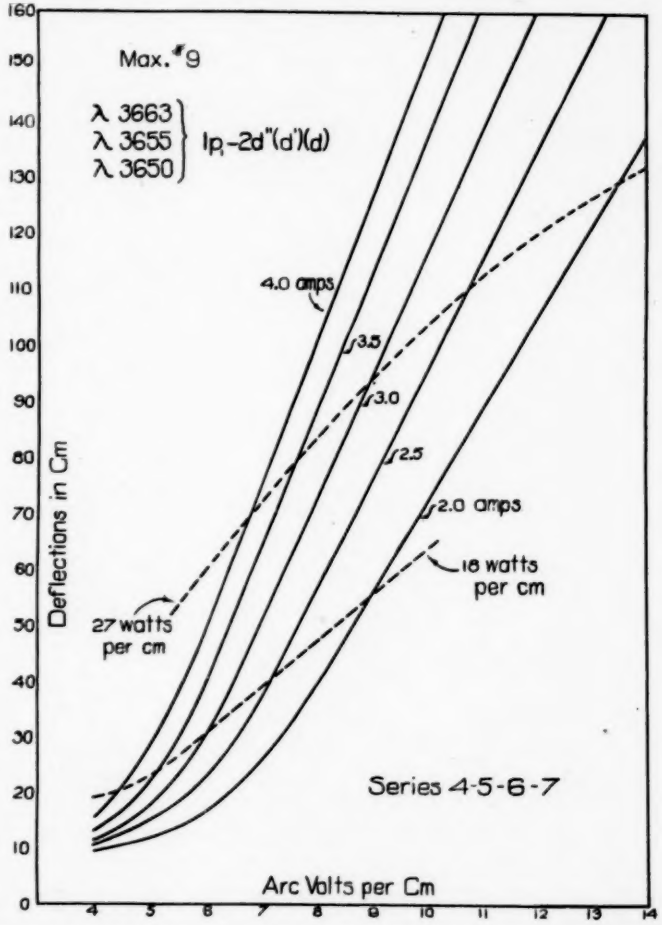


FIG. 6. *A family of curves showing the variation of the intensity of maximum No. 9 (3660 A) with voltage gradient, for different current values.*

The dotted curves connect points for the same power input.

are changed. The current has been plotted against galvanometer deflections for ten of the principal maxima, each of which is marked

with the most important wave-length in its group. The curves in general have similar shapes and may be compared directly, all being taken under the same conditions. The point at 2.5 amperes lies below practically every curve, and although no error in calculation can be

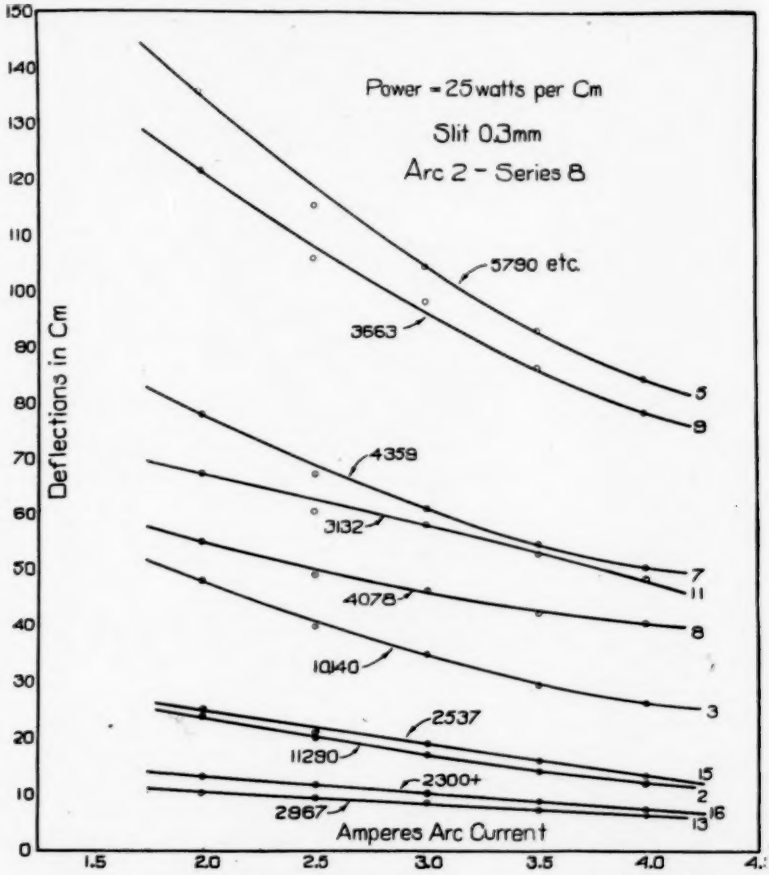


FIG. 7. *Curves showing the variation of energy in the most important maxima with current, at a constant power input of 25 watts per cm.*

found, apparently either the voltage or arc length was read incorrectly, since other results confirm the general shape of the curves as plotted. Lines 3 and 11 are somewhat anomalous; the first is the infrared group near 10140, and since the maxima were not corrected for background,

the increasing temperature radiation from the quartz is evidently the cause of the increase in slope. No explanation can be found for the anomalous curvature of the graph of maximum 11, and this point is being investigated further.

The figure shows at once the great gain in efficiency to be expected from using high voltages and low currents for a given power input; in other words, within the limits investigated, (0-2 atmospheres) the higher the pressure the better. Whether this holds to very high values is being investigated in the especially constructed high pressure arc mentioned before. It is evident that at 25 watts per cm and 4.5 amps., only one-half as much energy will go into the yellow-green maximum as if the current were 1.5 amps. and the voltage three times as great. Thus the arc, for high efficiency, should be cooled as little as possible.

Since the preceding curves all indicate that the highest efficiencies, within the limits investigated, are to be reached by building up pressure and voltage gradient as high as possible by keeping the rate of cooling at a minimum, it is important to see how far a further pressure increase will result in improved operation. The exact value of the best current would undoubtedly vary with the cross-section of the arc tube, due not so much to the change in current density as to the changed temperature gradient from walls to center. We may conclude from the results already obtained, however, that the average arc will show the least change in energy distribution for a given variation in total intensity if the pressure is as high as possible. Since this condition is also the one for greatest intensity of all maxima, and for greatest efficiency, the best operating conditions seem to be those with as high a pressure and voltage drop as is consistent with the life of the lamp and its mechanical structure.

We are continuing the study of the effect of pressure on the other variables, and hope to be able to reach pressures up to four or five atmospheres with an arc of different construction. After finally determining the best form and rating of arc to be used in photochemcial work we intend studying the degree of constancy in energy distribution to be expected from it, and the number of ergs per second per square mm obtainable in each wave-length band chosen for use.

In conclusion, we wish to thank the Cyrus M. Warren Fund for a grant covering the cost of building the special arc used.

JEFFERSON PHYSICAL LABORATORY (G. R. H.) AND
CHEMICAL LABORATORY OF HARVARD COLLEGE (G. S. F.)
HARVARD UNIVERSITY.
OCTOBER 21, 1924.

Study of the Discharge with Rising Characteristic which Precedes the Glow Discharge.—When the voltage across a glow discharge is plotted versus the current, it is seen to decrease as the current increases, the glow discharge having what is termed a "falling characteristic"; but if the P.D. between the electrodes be gradually increased from zero, the current at first remains imperceptible up to a certain voltage, then rises over an additional voltage range before the decline begins; Oelkers' observations are made upon this rising-characteristic discharge. He worked with electrodes 2 cm in diameter, 2 mm apart, mounted in the center of a tube 8 cm in diameter, filled with hydrogen or with air at pressures of the order of 6 mm; he wished above all things to avoid vapors from stopcock-grease, and consequently avoided using grease at all the joints except one, which was precisely the one where the grease would make the most trouble within the discharge region. Currents as low as 10^{-11} amp. were detected. The rising characteristic was followed in some instances up to the point where the falling characteristic commences; this point depends on the series resistance and hence cannot be regarded as a natural constant. It is interesting that in one instance (using hydrogen) Oelkers observed the current spontaneously rising at this point from .055 to .135 milliampere while the P.D. across the gap diminished from 475 to 403 volts, the process taking several minutes so that pairs of simultaneous readings on voltage and current could be taken at several stages during it. Most of the observations, however, relate to the fluctuations in the current; when working with very low currents and without series resistance in the circuit, Oelkers observed that on adding 2 volts to the battery emf he always got a sudden sharp rise in the current, which subsequently died away to a new equilibrium value in a period of 3 to 5 minutes—as if the effect of increasing the voltage across the gap had been to tear away some obstruction (the well known double layer is of course suggested) which directly began to re-form itself in a new fashion appropriate to the new value of voltage. When he subtracted 2 volts from the battery voltage, the current suddenly fell nearly or quite to nothing, and then gradually rose again to a new equilibrium value. With higher currents and a series resistance, these sudden sharp rises in current occurred irregularly and spontaneously, and each was followed by an interval in which the current was almost imperceptibly small. There are hysteresis effects, which after the fashion of all hysteresis effects are difficult to visualize. These observations refer to aluminium electrodes. With electrodes of copper, steel or silver the sudden sharp rises of current are absent except at high currents; but *en revanche* the current is never quite steady. But for a given metal, the behavior is about the same in hydrogen as in air. The discharge begins to be faintly visible to the eye at a current of about 5.10^{-10} amp.—[K. Oelkers, Halle; Ann. d. Phys. *74*, pp. 703-721; 1924.]

KARL K. DARROW

OPTICAL PYROMETERS*

W. E. Forsythe

Temperatures are determined with optical pyrometers by comparing the brightness of the source being investigated with that of the standard source. Optical pyrometers[1] must, therefore, have some means of making this comparison, either using the total light or that of a particular part of the spectrum.

Types of Optical Pyrometers

Instruments employing the total visible spectrum[2] are but little used at present. If any great range of temperatures is to be measured, there is difficulty due to the color difference between the comparison source and the source studied so that different observers differ widely in their readings.

Pyrometers using monochromatic light are divided into three classes,[3] the division being due either to the method of introducing the comparison source or to the method of making the comparison between this and the source being studied. The first class has its comparison source off at one side from the main telescope and the image of this comparison source is reflected into the eye-piece by a mirror located at the focus of the objective lens. Brightness matches are made either by varying the comparison source or by varying the brightness of the image of the source studied. Changes of intensity of the comparison source, which generally consists of an incandescent lamp filament, are made by varying the current through it or by varying an absorption device between the comparison source and the mirror.

In class two, called the polarization type,[4] an image of the source studied and that of the comparison source are observed through the same telescope where the light from the two sources is polarized in planes at right angles to each other. The balance is made by rotating a

* One of a series of reports prepared at the request of the Committee on Research Methods and Technique appointed by the Division of Physical Sciences of the National Research Council. It is the purpose of these reports to present in brief outline a summary, more or less critical, of the methods which may be employed in various kinds of physical measurements. See Editorial in this Journal *9* p. 410; 1924.

[1] Burgess: High Temperature Measurement,. p 291.
Glazebrook: Dict of Applied Physics, *1*, p. 643.

[2] Langmuir: Phys Rev., N.S. *2*, p. 138; 1915.

[3] Forsythe, J.O.S.A , *6*, 496; 1921.

[4] Foote, Fairchild, and Harrison: Tech. Paper, B.S. No. 170, p. 100.

nicol in the eye-piece. By operating the comparison source at a constant temperature, the instrument may be made direct reading by having a scale attached to the rotating nicol.

The two forms just described are troublesome to use because both require a rather large source for observation and, in general, it is not possible to see clearly the source being studied.

The third class,[5,6] called the disappearing filament type, consists of a telescope with a small incandescent lamp at the focus of the objective lens. A diagram of this pyrometer is shown in Fig. 1. The observer looks directly through the telescope and sees an image of the source studied crossed by the filament of the pyrometer lamp.

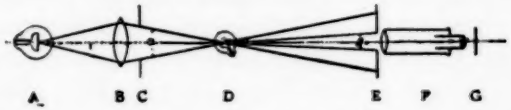


FIG. 1. *Diagram of arrangements on disappearing-filament optical pyrometer A—Background. B—Objective lens. C—Entrance cone diaphragm. D—Pyrometer filament. E—Eyepiece diaphragm. F—Eyepiece. G—Monochromatic filter.*

There are two types of disappearing filament optical pyrometers. In the first the photometric balances are made by varying the current through the pyrometer filament. In the second[7] this filament is kept at a constant current and thus at a constant brightness and balances are made by rotating a wedge-shaped circular absorbing screen between the pyrometer lamp and the objective lens, thus varying the apparent brightness of the image of the source studied. The advantage of this form is that it can be made direct reading, having a scale attached to the frame that carries the specially made absorbing screen.

As the disappearing filament pyrometer is generally constructed the object studied is seen upside down the same as in the ordinary telescope. A new instrument that has recently been examined has an additional lens or special prism in the eyepiece so that the source observed is seen in its proper position. For certain work this may be advantageous.

Good results can be obtained with any of the above forms of optical pyrometers. However, if an experimenter wishes to build one himself, the best is the first form of the disappearing filament type described above. All that is needed is an objective lens about one and one-half

[5] Waidner & Burgess: Bull. B.S. *6*, p. 189; 1909.
[6] Mendenhall: Phys. Rev. *33*, p. 74; 1911.
[7] Foote and Fairchild: A.I.M.M.E. Symposium, p. 331; 1920.

inches in diameter having a focal length of about six to eight inches, a small lamp to be used as a pyrometer lamp, a low power eye-piece with a focal length of about two to three inches, two pieces of red glass, a small battery, a resistance to vary the current through the pyrometer lamp, and an ammeter to measure this current.

In an optical pyrometer the monochromatic beam may be obtained[6,8] by using some form of spectrometer or by the use of a monochromatic glass filter[9] in the eye-piece. If, for any reason, it is desired to study any very definite part of the spectrum, it may be necessary to use a spectroscope as the eye-piece, but in general a monochromatic screen is as good, if not better, than the spectroscope. A diagram of one form of the spectral pyrometer is shown in Fig. 2. Glass screens may be obtained[10]

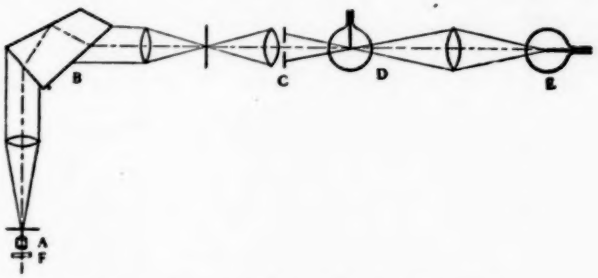


FIG. 2. *Diagrammatic sketch of spectral pyrometer.*

that will give a sufficiently monochromatic beam in either the red, green, or blue part of the spectrum. For pyrometric work it is in general better to use a red screen for the following reasons. In the first place, at low temperatures, the red radiation first becomes visible and thus readings may be made at a somewhat lower temperature with the red glass. In the second place, the color change for a change in wavelength[11] is much less in the red part of the spectrum than in the green, which makes the red the better part of the spectrum for this work. The third reason for using red glass is that better monochromatism can be obtained in glass of this color. The fourth advantage of the red screen is that when measuring a temperature so high that a rotating sector[6] or

[8] Henning: ZS. f. Instru., *30*, p. 61; 1910.

[9] Forsythe: J.O.S.A. *5*, p. 85; 1921.

[10] Good glass screens of various color can be obtained from the Corning Glass Works at Corning, N. Y. The set of Wratten and Wainwright screens made by the Eastman Kodak Co. will be found very useful.

[11] Jones: J. Frank. Inst., *183*, p. 500; 1917.

absorbing glass is necessary, the transmission of the sector or absorbing glass that must be used to reach a particular temperature is greater for the longer wave-length than it is for the short wave-length.

In Fig. 3 is shown the spectral transmission of a number of red screens that are useful for this class of work. However, it is sometimes necessary to study other parts of the spectrum with the pyrometer. In Fig. 4 are shown the spectral transmissions of a red, a green, and a blue glass, that are quite satisfactory for this kind of work.[10]

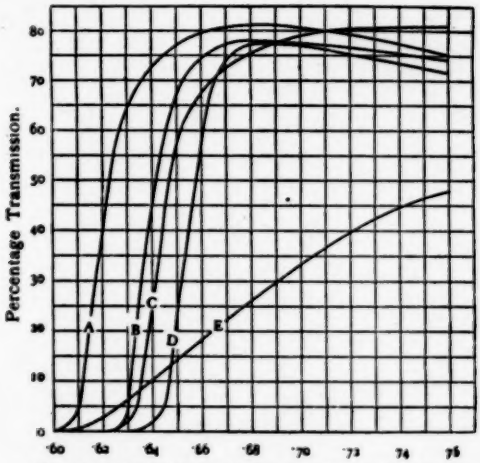


FIG. 3. *Spectral transmission of various red glasses.*
Curve C for Jena red 4512, 2.93 mm thick.
Curve E for Jena red 2745, 3.2 mm thick.
Curve A for Corning high transmission red, marked 150 per cent, 5 mm thick.
Curve B for Corning high transmission red, marked 50 per cent, 5 mm thick.
Curve D for Corning high transmission red, marked 28 per cent, 6 mm thick.

EFFECTIVE WAVE-LENGTH AND MONOCHROMATIC SCREENS

An optical pyrometer can be calibrated and so used as to make unnecessary a knowledge of the extent to which the screen is monochromatic but if calibrated from a black body furnace held at one temperature as described below,[6] this knowledge is necessary. The effective wave-length[12] also must be known if the pyrometer is used to measure the temperature of non-black bodies. In using the pyrometer, the integral luminosities through the monochromatic screen are compared, for

[12] Hyde, Cady, Forsythe: Astrophys. J, *42*, p. 294; 1915.

which reason the effective wave-length of the screen corresponding to a certain temperature interval has been defined as the wave-length for that temperature interval of a black body for which the ratio of its radiation intensities equals the ratio of the integral luminosities through the screen used.

Knowing the spectral transmission of the glass, it is possible to calculate the effective wave-length for any temperature interval by means of the following equation:

$$\left.\frac{\int_0^\infty J(\lambda T_1)\, V_\lambda t_R d\lambda}{\int_0^\infty J(\lambda T_2)\, V_\lambda t_R d\lambda} = \frac{J(\lambda T_1)}{J(\lambda T_2)}\right]_{\lambda_e} \qquad (1)$$

where t_R is the spectral transmission of the glass and V_λ is the visibility and $J(\lambda T)$ the energy distribution (Wien's equation). These integrals can be computed by the step-by-step method with sufficient accuracy for this purpose.

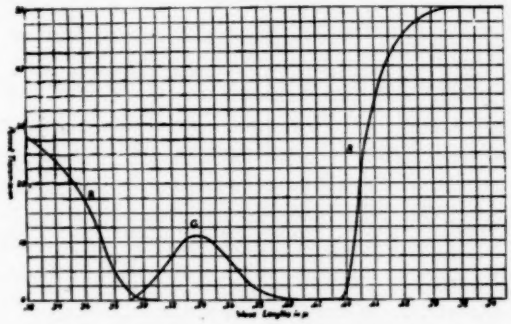


FIG. 4. *Spectral transmission of different screens.*
Curve B, two thicknesses blue uviol glass, total thickness 3.9mm.
Curve G, two thicknesses green glass, total thickness 5.2 mm.
Curve R, two thicknesses Jena red glass, total thickness 6.8 mm.

Using this equation, the effective wave-length of the red glass the spectral transmission of which is shown in curve *C*, Fig. 3 has been calculated[13] for a number of temperatures and curves plotted as shown in Fig. 5. From these curves the effective wave-length can be obtained for any temperature interval within the range. To show how these curves may be used, the effective wave-length for a couple of temperature intervals will be found. The effective wave-length between 1800°K and 2900°K is given by the ordinate of the point where the 1800°K

[13] Forsythe: J.O.S.A., *4*, p. 322; 1920.

curve crosses the 2900°K ordinate; that is, it is 0.6644μ. For the range between 2100°K and 2900°K the effective wave-length is likewise given by the point where the 2100°K curve would cross the 2900°K ordinate. However, in this case the 2100°K curve is not drawn, but will have to be imagined as drawn parallel to the 1800°K, one point of its position being determined by where the curve *C* crosses the 2100°K ordinate. The effective wave-length for this interval is 0.6641μ. It can be seen from the figure that the effective wave-length for any temperature interval is given quite closely by the mean of the limiting effective wave-length for the two temperatures.

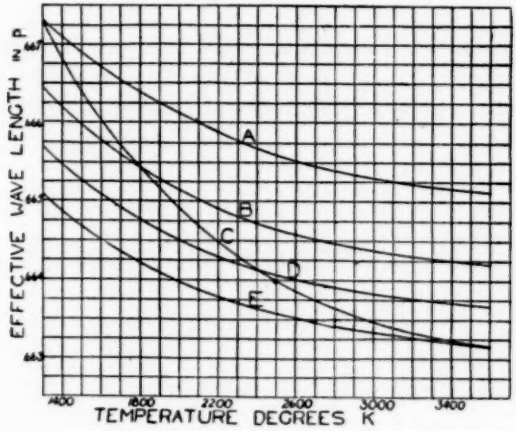


FIG. 5. *Effective wave-length for Jena red glass.*

Curve A, effective wave-length from 1300°K to other temperatures.
Curve B, effective wave-length from 1800 to other temperatures.
Curve D, effective wave-length from 2400 to other temperatures.
Curve E, effective wave-length from 3600 to other temperatures.
Curve C, limiting effective wave-length.

CALIBRATION

An optical pyrometer is calibrated[14] by finding the relation between the temperature of the standard source and the readings of the scale, whether it be the one engraved on the direct reading instruments or the current through the pyrometer filament for the first form of the disappearing filament pyrometer described above. Direct reading instruments are generally so constructed that if the comparison source

[14] Forsythe: Chem. & Met. Eng. *22*, p. 1211; 1920.

is at the right brightness for one temperature, the instrument will give the correct values for other temperatures.

The direct method of calibrating a pyrometer is to have a black body that can be operated at various known temperatures over the entire range desired. But it is very difficult in the first place to get a black body that can be operated at sufficiently high temperatures for this purpose, and the determination of these temperatures presents added difficulties. Standard melting points are used as points of reference but there are not enough standard melting points in the range of the pyrometer to give a satisfactory calibration, so methods have been devised for calibrating the pyrometer from a single standard point.[6]

In such a calibration with the furnace held at a definite melting point temperature, it is necessary to have some means such as a rotating sectored disk or an absorbing glass of known transmission for cutting down the apparent brightness of the incident radiation. If readings are made of the current through the pyrometer filament for an apparent brightness match using a sector or an absorbing glass between the pyrometer lamp and the standard black body, there will be obtained a measure, in terms of the current through the pyrometer filament, of a brightness that is some known fraction of that of the standard black body at the standard temperature. If monochromatic radiation is used, it is easy to calculate by the following formula derived from Wien's equation the temperature, T_2, of the black body corresponding to this measured brightness; that is, to this current through the pyrometer filament, from T_1, the standard temperature:

$$\log R = \frac{c_2 \log e}{\lambda}\left[\frac{1}{T_2} - \frac{1}{T_1}\right] \qquad (2)$$

where the temperatures, T_1 and T_2, are on the absolute centigrade scale, R is the transmission of the sector, and λ the wave-length used. If the measurements are made with a red glass in the eye-piece, the temperature that would correspond to this fraction of the brightness of the standard black body can be calculated just as before, except that in this case the effective wave-length[12] of the red glass for this temperature interval is to be used. By using a number of sectors[6] or absorbing glasses with different transmissions, points can be obtained for a curve showing the relation between the temperature and current.

In column 3 of Table 1 are given a number of apparent temperatures each of which corresponds to the red brightness obtained when a rotating sector having the transmission given in column 1 is used between

the pyrometer filament and a black body at the temperature of melting palladium. The red glass used in this computation is Jena 4512, having the transmission shown in Fig. 3, curve *C*. The effective wave-lengths for the different intervals are given in column 2. If a different red glass is used, different values of the temperatures will of course be obtained.

To measure temperatures higher than it is safe to operate the comparison source, place a rotating sector[1,6] or absorbing screen between the pyrometer lamp and the source the temperature of which is being

TABLE 1. *Temperatures corresponding to different percentages of the radiation from a black body held at the temperature of melting palladium (1829°K) using a red glass with an effective wave-length which varies as is shown in column two.* ($c_2 = 14330\mu$ deg.)

Transmission of Sector	λ_e	T_2 in Degrees Kelvin
0.749	0.6654	1786
.499	.6655	1728
.2443	.6656	1632
.0830	.6658	1509
.0336	.6660	1426
.01668	.6662	1356
.00542	.6665	1267

measured, and make a brightness match. The ratio of the brightness thus measured to that corresponding to the pyrometer calibration is given by the reciprocal of the transmission of the sector or absorbing screen used. The temperature for this brightness is to be calculated from the temperature, corresponding to the current through the pyrometer filament, and the transmission of the sectored disk or absorbing glass used by means of equation 2, except that for this computation R is the reciprocal of the transmission of the sector or glass; and as a consequence T_2 will be greater than T_1. A very convenient method is to work out such extrapolations for the various sectors for a number of temperatures and to plot these extrapolations against the initial temperatures so that the extrapolated temperatures can be obtained corresponding to any pyrometer reading when a particular sector or absorbing screen is used.

In Table 2 is given a series of such extrapolated temperatures using different sectored disks. In column 1 are given the initial temperatures; that is, the temperatures that would correspond to the pyrometer reading without any sector. In columns 3, 5, 7, and 9 are given the extrapolated temperatures corresponding to the same pyrometer readings when the sector, the transmission of which is given at the head of the column, is used between the pyrometer lamp and the source

studied, and a red glass having the effective wave-length shown in columns 2, 4, 6, and 8 is used in the eye-piece. This red glass is the one, the spectral transmission of which is shown by curve *C*, Fig. 3. For a different red glass of course different values will be obtained. The sectors with the transmission given in the table or sectors which transmit 1/10, 1/60, and 1/120 of the incident radiation are very satisfactory for this work.

TABLE 2. *Extrapolated temperatures, using red glass, for various sectors with transmission as given.* $c_2 = 14330\mu$ *deg.*

Transmission of Sector		0.2443	0.0830		0.01664		0.00542	
Initial Temperature	λ_e	Deg. K	λ_e	Deg. K	λ_e	Deg. K	λ_e	Deg. K
1200°K	0.6675μ	1303	0.6673μ	1398	0.6670μ	1555	0.6668μ	1693
1300	.6671	1421	.6668	1530	.6665	1727	.6663	1899
1400	.6667	1541	.6664	1672	.6661	1909	.6659	2119
1500	.6663	1663	.6660	1816	.6657	2100	.6654	2355
1600	.6659	1788	.6656	1964	.6652	2300	.6650	2611
1700	.6655	1913	.6652	2117	.6648	2512	.6646	2888
1800	.6651	2042	.6648	2274	.6645	2739	.6643	3188
1900	.6647	2171	.6645	2435	.6642	2968	.6640	3515

For the highest accuracy it is necessary to take account of the variation in the effective wave-length for different temperature intensities. To show just what errors would result Table 3 has been prepared. In column one are given the temperatures extrapolated from the palladium point; in column two the effective wave-length for the interval between the palladium point and the temperature given in column one. In column three is given the error that would result in the use of the effective wave-length for the interval between the gold point and the palladium point rather than the proper wave-length given in column two. It will be noted that the error is negligible excepting for extreme ranges.

It will also be noted that the error in the effective wave-length is small being only about one-third of one per cent for the maximum difference given in the Table. If an effective wave-length one per cent longer than the proper effective wave-length is used, the errors given in column four would result.

Wien's equation is generally used to calculate temperatures from brightness measurements rather than Planck's more exact form because it is easier to compute with this form, and it gives results that are

sufficiently accurate for temperatures that are more commonly studied. Table 4 has been prepared to show just what differences may be expected. As can be seen from the table the error will be very small as long as the product $\lambda T < 2400$.

TABLE 3.—*Errors in extrapolated temperatures due to an error in the effective wave-length*

Temperature Extrapolated from 1829°K	Correct Effective Wave-Length	Error due to Using Wrong Effective Wave-Length	
		A*	B**
1000	0.6675	+0.7	−4.7
1336	0.6664	.0	−3.6
1600	0.6657	−0.2	−2.0
2000	0.6651	+0.4	+1.9
2500	0.6647	+2.3	+9.2
3000	0.6645	+5.4	+19.3

* A—using effective wave-length for interval between 1336 and 1829°K.
** B—using effective wave-length that is 1% too large.

A rotating sector properly calibrated is probably the best means of reducing the observed brightness of a source. A rotating sector disk can be calibrated by actually measuring the size of the openings on a circular dividing engine or it may be calibrated photometrically.[15]

TABLE 4.—*Energy Ratios and Corrections to Temperatures as Calculated with Wien's Equation to Reduce Them to What Would be Obtained from Planck's Equation for $\lambda = .665\mu$ for Definite Values of Brightness.*

Temperature in °K	λT	Energy Ratios E_v/E_w	Corrections to True Temperature
2000	1330	1.00003	−<0.01°K
2600	1730	1.0003	− 0.05
3000	1990	1.0007	− 0.3
3600	2390	1.0025	− 1.5
4500	2990	1.0084	− 8.5
5000	3320	1.0135	− 15.6
6000	3990	1.0280	− 47.0

The speed of rotation of the sector should be such that no flicker is noticeable. If the sector rotates so that there will be about thirty dark and bright alternations per second, no flicker will be noticed for sectors where the open part is about equal to the closed part. For very small

[15] Cady: Trans. I.E.S., *16*, p. 138; 1921.

openings as for instance a two degree sector with two one-degree openings, the speed must be such as to give 60 to 80 alternations per second. The rotating sector must be located near the pyrometer lamp in order to avoid errors due to diffraction.[16] Although a rotating sector may be better to use than an absorbing screen there are times when such a screen must be used.

"NEUTRAL" ABSORBING SCREENS

In using an absorbing screen with an optical pyrometer some care is necessary. Unfortunately it is not easy to get an absorbing screen that is a neutral tint, that is, one that has the same transmission for different parts of the spectrum. Absorbing screens that are not neutral tint have a total transmission that depends upon the spectral distribution of energy from the source studied. The transmission for different conditions can be measured by comparison with sectored disks. This is sometimes very difficult since it is necessary to have a source with known distribution. For this a calibrated tungsten lamp can be used, obtaining the distribution from the color temperature.[17] A more convenient method is to measure the spectral transmission of the screen and then calculate the total transmission for the different temperatures of the source for which it is to be used. The total transmission T_A of an absorbing glass when used with a red glass (or any other color) is given by the following equation:

$$T_A = \frac{\int_0^\infty J(\lambda T)\, V_\lambda t_R t_A\, d\lambda}{\int_0^\infty J(\lambda T) V_\lambda t_R d\lambda} \qquad (3)$$

where $J(\lambda T)$ (Wien's equation) is the energy distribution, V_λ the visibility, t_R the spectral transmission of the red glass and t_A the spectral transmission of the absorbing glass. This integral can be calculated by the step-by-step method with sufficient accuracy for this purpose. This is done by forming the product as indicated by the equation for a number of different wave-length intervals and then taking their sum. This of course gives the area under a stair step curve but by making the step sufficiently small ($.005\mu$), a very close approximation can be obtained. The values of the different transmissions and the visibility[27] for the different wave-lengths are to be obtained from a smooth curve drawn through the experimental points. Sometimes it is desired to use an absorbing screen with so small a total transmission that it is difficult

[16] Hyde, Cady, Forsythe: Astrophys. J., *42*, p. 303; 1915.
[17] Hyde: A.I.M.M.E. Symposium, p. 285; 1920.

to measure the spectral transmission. In such cases a very good plan is to use two or more screens and measure their separate spectral transmissions and then obtain their combined spectral transmission which at any wave-length is given by the product of their separate transmissions for this wave-length. It is to be noted that the combined total transmission of two absorbing screens is not in general equal to the product of the separate total transmissions.

In Fig. 6 is shown the spectral transmission of a number of absorbing screens that are suitable for this kind of work. In Fig. 7 is shown the total transmission of two of the absorbing screens as a function of the temperature of the source studied. In calculating an extrapolated temperature for a particular absorbing screen the transmission of the screen to use is the one corresponding to the temperature that is

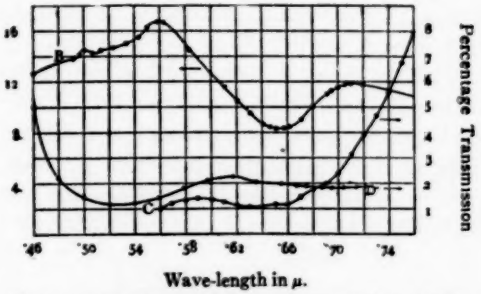


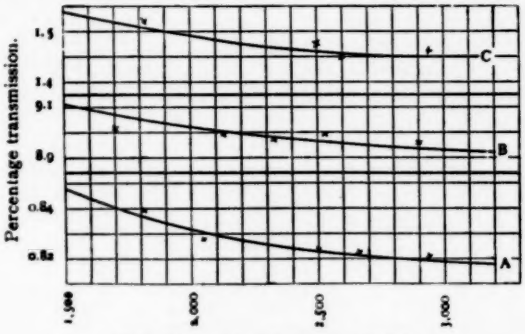
FIG. 6. *Spectral transmission of various absorbing glasses.*
Curve B—Jena absorbing glass 1.5 mm thick.
C—Noviweld obtained from Corning Glass Works. Shade about 6.
D—Leeds and Northrup absorbing glass made of purple and green glass.

measured. This may require a second approximation. The total transmission of absorbing screens also depends upon the temperature of the screens themselves. In Table 5 are given the total transmissions for two different temperatures of the glasses of two of the screens the spectral transmissions of which are given in Fig. 6. When absorbing

TABLE 5.—*Transmission of the absorbing glasses at different temperatures.*

T Temperature of glass	Per cent transmission of Noviweld absorbing glass	Per cent transmission of Jena absorbing glass
20°C	1.70%	8.96%
102	1.55	8.90
200	1.39	8.87

screens are used they must be mounted so that they can always be put back in exactly the same position, since the transmission is apt to be different in different parts of the screen. The noviweld screens which can be obtained in different shades from the F. W. Hardy Company of Chicago are very satisfactory for this purpose. The absorbing screens made by the Eastman Kodak Co. are also very good.



Color Temperature in Degrees Kelvin.

FIG. 7. *Total transmission of absorbing glasses, as a function of temperature when used with red glass 4,512, 5.8 mm thick.*
Curve A—Two pieces Jena absorbing glass.
Curve B—One piece Jena absorbing glass.
Curve C—Noviweld glass from Corning Glass Works.

In connection with some special work[18] it was necessary to extend the calibration of an optical pyrometer to measure temperatures up to about 6000°K. This was done by using two absorbing glasses between the pyrometer lamp and the source being investigated. The spectral transmissions of the two glasses are shown by curves *A* and *B*, Fig. 8, and that of the combination is shown by curve *C*, Fig. 8. Curve *C* was obtained from the product of the spectral transmissions of the two glasses at the different wave-lengths.

The transmission of either of these glasses alone was readily measured while a single absorbing glass with a transmission equal to that of the combination of the two would have been very difficult to measure. The total transmissions of these absorbing glasses for red radiation were calculated from their spectral transmissions by equation 3 for various temperatures of incident radiation and are shown in Fig. 9. From the values of the total transmissions thus obtained, the extrapolated

[18] This calibration was made for Mr. McKay of the Research Laboratory of the General Electric Co. at Schenectady.

temperatures for one glass and for the combination of the two were calculated by means of equation 2 and are shown in Table 6. It will be seen that when the two glasses are used together, temperatures may be measured up to about 6000°K.

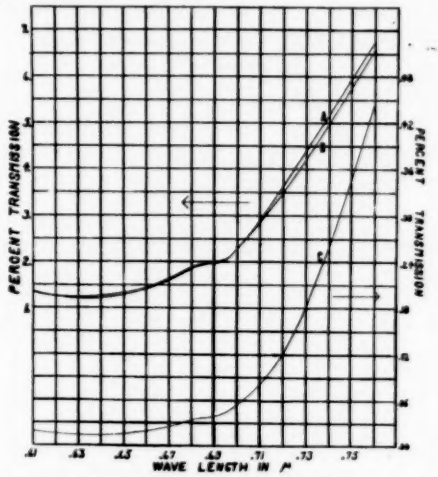


FIG. 8. *Spectral transmission of two absorbing screens (Curves A & B), and the spectral transmission of the two together (Curve C).*

These values of extrapolated temperatures were calculated by means of Wien's equation, and are in error by the amounts shown in Table 4. This error is negligible except at the very high temperatures.

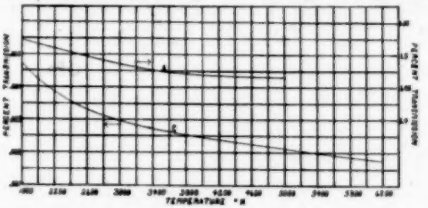


FIG. 9. *Total transmission for red radiation of the absorbing screen whose spectral transmission is shown by curve A, Fig. 8, and of two absorbing glasses whose spectral transmission is shown by curve C, Fig. 8.*

To operate a black body as a standard for the calibration of an optical pyrometer, requires considerable time and some very special equipment. Fortunately, excepting for the most accurate work, this is unnecessary. Two methods are available that have been found to give accurate results and are simple and easy to operate.

The first method consists of making a black body by folding a thin strip of some metal (i.e. gold, palladium or platinum) into the form of a narrow wedge or by forming a hollow cylinder with a very small hole through the side (Worthing).[19] Mendenhall[20] has shown that the radiation coming from the inside of the wedge of a polished metal such as platinum is almost 100 per cent black body radiation for a 10° wedge. This device has been further worked out by Spence[21] as a standard for the calibration of optical pyrometers. In Fig. 10 is shown a diagram of this device. Mendenhall obtained his temperature by melting these standard metals when in the form of a wedge. Spence has shown how thermocouples can be used to determine the temperatures for the purpose of calibrating the pyrometer.

TABLE 6.—*Extrapolated temperatures using red glass for one absorbing glass and for two absorbing glasses having the transmissions shown in Figs. 8 and 9.* $c_2 = 14330\mu$ deg.

Initial Temperature	Transmission of one absorbing glass	Extrapolation temperature for one absorbing glass	Transmission of two absorbing glasses	Extrapolated temperature for two absorbing glasses
1200°K			.000238	2229
1300			.000233	2606
1400	.0149	1922	.000230	3047
1500	.0148	2116	.000227	3568
1600	.0147	2322	.000224	4149
1700	.0147	2540	.000221	4972
1800	.0146	2771	.000218	5952

The second method is to use a tungsten lamp with either a wire or ribbon filament, this filament to be calibrated as a standard of brightness. It is very convenient to have the filament calibrated so that the currents are known for the brightnesses as observed through a definite monochromatic glass for the black body at the temperature of melting gold and melting palladium.

It has been found that such a lamp, if not operated at a higher temperature than that necessary to give a red brightness the same as that of a black body at the palladium point, will have a very long life. Such standard lamps are very valuable for a hurried check on the calibration of an optical pyrometer since they can be calibrated at several points over the range desired to calibrate the pyrometer lamp. Such

[19] Worthing: Phys. Rev., N. S. 10, 377, 1917.
[20] Mendenhall: Astrophys. J. *33*, p. 91, 1911.
[21] Spence: Astrophys. J., *37*, p. 194; 1913.

lamps can be obtained with wire filaments up to 20 mils (.5 mm) in diameter and with ribbon filaments up to 3 or 4 mm in width. A 20 mil filament suitable for such work requires about 15 amperes and 3.7 volts for a brightness equal to that of a black body at the palladium point and a 2 mm ribbon lamp requires a current of 16 amp and 2.5 volts for the same brightness. The exact point of the filament to be used as the standard is indicated by a bend in the wire filaments and by a small notch in the edge of the ribbon filaments.

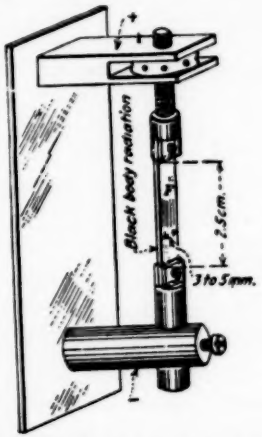


FIG. 10. *Wedge opening used by Mendenhall for true-temperature measurements.*

PYROMETER LAMPS

Some care is needed in selecting a pyrometer filament. Carbon filaments are quite satisfactory for low temperature, but they will not have a long life if operated at a very high temperature. Tungsten filaments are just as good as carbon and have the added advantage that they will have a life of several thousand hours, if they are not operated at a temperature higher than that necessary to match in brightness a black body at the temperature of melting palladium. Tungsten pyrometer lamps may be operated at a temperature so that they will match in brightness a black body at a temperature of 2000°K, but they will not be as reliable as standards. At this temperature they should have a useful life of a few hundred hours. The tungsten filament is often constructed with a small bend at the exact point where the filament is to be observed. A small pointer is sometimes used to help locate the exact spot. A filament with a diameter of 2-1/2 mils, (.06 mm)

has been found quite satisfactory for most work. Sometimes a larger or smaller filament may be desired. The wire for the filament should be as uniform as possible so that it will heat to uniform brightness. It is also well to have its surface free from scratches, since the scratches may make the filament appear non-uniform in brightness. The filament should be three or four centimeters long so that the end cooling will not have any great effect on the temperature at the spot used as a point of reference. All joints where the filament is fastened to the leads should be welded and the lamps should have as good vacuums as possible. Before a new pyrometer lamp is standardized it should be aged. A satisfactory aging is to burn a 2-1/2 mil filament for twenty-four hours on 0.75 ampere. The current to use for filaments of different diameters varies as the 3/2 power of the radius.

The bulb should be very carefully selected so that a clear image of the background can be seen through it. There should be no imperfections in line with the point where the comparison of brightnesses are to be made. Distortions due to the bulb will be less troublesome in larger bulbs than in smaller ones. In general, a spherical bulb about 2-1/2 inches (6 cms) in diameter will allow a satisfactory view of the background to be obtained. Where the highest accuracy is desired it may at times be desirable to use a pyrometer bulb having plain glass windows which can be made after the method described by Fairchild.[22]

OBSERVATIONS ON NON-BLACK SOURCES

Optical pyrometers are generally used to measure the temperature of furnaces where approximately black body conditions exist. However, it is often convenient to study non-black bodies with an optical pyrometer and to assign a temperature to a brightness of a non-black body as if it were a black body. Since all non-black bodies are in general less bright than a black body at the same temperature, such temperatures are less than the true temperatures. They are called black body brightness temperatures[17] or shortened to brightness temperatures. When we say a non-black body has a brightness temperature of 1500°K for $\lambda = .665\mu$ we mean that it has the same brightness as a black body at 1500°K for this wave-length interval.

For any source with known emissivity, true temperatures can be obtained from brightness temperatures. By emissivity of a particular source is meant the ratio of the brightness for a particular wave-length interval of the source to that of the black body for the same wave-

[22] Fairchild: J.O.S.A., *4*, p. 496; 1920.

length interval and at the same temperature. The emissivity may vary with the wave-length as well as with the temperature. Thus, the brightness temperature will in general depend upon the wave-length used. If strictly monochromatic radiation is used there is no uncertainty as to the wave-length to which the brightness temperature is to be assigned. However, if a so-called monochromatic screen is used which in reality has a rather wide transmission band, this wave-length is not directly evident. For this latter case, it can be shown[23] that the brightness temperature is to be ascribed to the effective wave-length of the screen for the interval between a temperature corresponding to the brightness temperature and one corresponding to the color temperature[17] of the source studied.

The relation between brightness and true temperatures as derived from Wien's equation is

$$\frac{1}{T} - \frac{1}{S_\lambda} = \frac{\lambda \log \epsilon}{c_2 \log e} \tag{4}$$

where T is the true temperature, S_λ the brightness temperature and ϵ the emissivity for wave-length λ.

As an example of emissivity the following may be mentioned. For platinum the emissivity for the red ($\lambda = .665\mu$) varies from .36 for a temperature of 1000°K to .32 for a temperature of 1400°K. For tungsten the variation for the red ($\lambda = .665\mu$) is from .46 for $T = 1200$°K to .42 for $T = 2800$°K. For the blue for tungsten ($\lambda = .467\mu$) the change is from .49 for $T = 1200$°K to .45 for $T = 2800$°K. The emissivity of carbon for the red ($\lambda = .665\mu$) varies from .86 for $T = 1200$ K to .79 for $T = 2200$°K. Iron oxide has a very high emissivity being about 95 per cent for the red radiation for $T = 1400$°K. The values of brightness temperature obtained by observations on heated iron oxide with an optical pyrometer are very nearly equal to the true temperature.

In Table 7 are given the values that must be added to different brightness temperatures for different values of the emissivity in order to obtain the true temperature.

When working with an optical pyrometer, care should be taken to have all the parts of the optical pyrometer in accurate alignment. There must be a fixed limiting diaphragm so that there will be a definite cone of rays entering the eye. This diaphragm is generally located somewhere between the pyrometer lamp and the eye-piece. If this

[23] Trans. Faraday Soc., *15*, p. 21; 1920.

definite cone is always filled and is of such size that all the light enters the eye, the brightness of the source studied will not depend[24] upon its distance from the pyrometer. The cone of rays from the objective lens must be somewhat larger than this[25] or there will be trouble due to diffraction at the pyrometer filament. Unless it is much larger,[26] there must be a fixed diaphragm between the objective lens and the pyrometer

TABLE 7. *Corrections to Add to Brightness Temperature Readings for Different Emissivity*

Emissivity	Pyrometer using red light, wave-length, $\lambda = 0.665\mu$ at observed temperatures, degree Kelvin of —															
	1000	1100	1200	1300	1400	1500	1600	1700	1800	2000	2200	2400	2600	2800	3000	3600
0.10	119	146	176	209	246	286	329	377	429	543	676	827	1000	1196	1416	2251
.20	80	98	118	140	163	189	217	247	279	352	433	524	627	740	867	1325
.30	59	72	86	102	119	137	157	178	201	251	308	371	442	519	604	907
.40	44	54	64	76	89	102	117	132	149	186	227	272	323	379	439	651
.50	33	40	48	57	66	76	87	98	110	137	167	201	237	277	320	472
.60	24	29	35	41	48	55	63	71	80	99	121	145	170	199	229	336
.70	17	20	24	29	33	38	43	49	55	68	83	99	117	136	157	228
.80	10	13	15	18	21	24	27	30	34	42	51	61	72	84	96	139
.85	7	9	11	13	15	17	19	22	25	31	37	44	52	60	70	96
.90	5	6	7	8	10	11	13	14	16	20	24	28	33	39	45	65
1.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The values given in this table also give the correction for a window having the transmission given in column 1 for different temperatures of the source when this window is used between the source and the pyrometer.

lamp or the brightness match obtained will depend upon the change of the angle[27] of the cone from the objective lens due to a motion of the objective lens to bring objects at different distances to a focus.

NELA RESEARCH LABORATORY,
NATIONAL LAMP WORKS,
CLEVELAND, OHIO,
MARCH 1924.

Theory of the Current-Voltage Relation in the Corona.—The corona discharge between a cylinder and a wire stretched along its axis is seen to be divided into a luminous sheath surrounding the wire and a dark region extending from the sheath to the cylinder. Fazel and Parsons conceive that ionization goes on in the luminous sheath, while in the dark region there is no ionization, but a procession of ions

[24] Schuster, "Theory of Optics," p. 152; 1909.
[25] Worthing and Forsythe: Phys. Rev., *4*, p. 163; 1914.
[26] Fairchild: J.O.S.A.& R.S.I., *7*, p. 543; 1923.
[27] Trans. I.E.S., *13*, p. 523; 1918.

of one sign moving radially towards the cylinder. Making the further assumption that the speed of the ions at any point is proportional to the field there, and treating this as a problem in space-charge theory, it is obvious that the assumptions suffice to determine the current-voltage relation, in terms of the dimensions of the system and the mobility of the ions, provided that the proper physical boundary conditions can be guessed. In this case, it is assumed that the field strength at the edge of the luminous sheath is always equal to the "critical" field strength, i. e. the value of field strength which is attained at the surface of the wire at the moment when corona commences. The situation is also complicated by the fact that the radius of the luminous sheath is known to vary with the voltage across the discharge. Assuming this variation to be linear, and making certain approximations, the authors arrive at a formula for the current-voltage relation which involves the radii of cylinder and wire, the critical voltage aforesaid, and two constants which depend on the mobility of the ions and the assumed relation between voltage and radius of sheath. The test of the theory lies (a) in whether values can be assigned to the two constants such that the formula will agree with observations, and (b) whether when this is done the values for the constants are compatible with a plausible value for the mobility of the ions. The current-voltage relation is determined experimentally for air, damp or dry, at atmospheric pressure and various temperatures between 290° and 417° K.; the test (a) is satisfied; as for (b) the values of mobility demanded by the choice of constants are generally from two to four times as great as the directly measured ones. The values chosen for the constants also fix the relation between voltage and radius of sheath, but no comparison between theory and experiment is made. [C. S. Fazel and S. R. Parsons, University of Illinois; Phys. Rev. (2) *23*, pp. 598-607, 1924.] K. K. DARROW

Polarized Fluorescence of Dye-solutions.—Simple experiments continuing the previous ones and confirming in general the rule that the degree of polarization increases with the viscosity of the liquid in which the dye is dissolved, but not beyond a certain limit generally of the order 35%. It is found that the degree of polarization varies little or not at all with concentration below a certain critical value of the concentration, but diminishes beyond; this diminution, while partly due to a process of primary fluorescent light exciting a secondary fluorescence which both consumes the primary fluorescent light and mixes the residuum with unpolarized secondary light, is probably not entirely explicable in this way. Similarly there is an effect of temperature which is probably not entirely an effect of changing viscosity; and in two cases the polarization seems to increase with the frequency of the exciting light (the range of wave lengths is small, and filters are used to modify the constitution of the exciting light). [W. L. Lewschin; ZS. f. Phys. *26*, pp. 274-283; 1924.] KARL K. DARROW

INSTRUMENT SECTION

THE COLOR CORRECTION OF AN ACHROMATIC DOUBLET

By T. Townsend Smith*

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In calculating lenses it is customary to "correct" for the chromatic aberration by making the focal length of a combination the same for two different wave lengths.[1] To correct for the spherical aberration it is customary to choose some intermediate wave length, and to choose such shapes for the lenses that the correction is good for that wave length.

For visual work in achromatizing, the wave lengths chosen are commonly those of the *C* and *F* lines, partly because this choice brings the minimum focal length of the combination in the part of the spectrum to which the eye is most sensitive, about 550mμ, and partly, probably, because the hydrogen tube provides a convenient source for these lines. In correcting for the spherical aberration, the light commonly used is the yellow sodium light, the *D* lines. The choice here is probably on the ground of the convenience of the source and of the high visual brightness of this region of the spectrum. To get rid of the somewhat objectionable sodium flame, it has been proposed to use the yellow helium line, which has nearly the same wave length as the sodium line.[2]

* This paper was prepared at the suggestion of Dr. I. C. Gardner, the chairman for the year 1923 of the Committee on Refractometry of the American Optical Society, and represents the author's contribution to the work of that committee.

[1] cf. Scheibner in Koenig. Sachsischen Ges. Wiss., p. 541; 1878.

[2] Harting, Zts. Instk. 1908.

The object of the present paper is to present a reasonable basis for the choice of wave lengths to be used in calculating lenses and in specifying optical glass.

The principal results of the investigation are:

1. An expression for the variation in focal length of an achromatic doublet in terms of the Hartmann interpolation formula.

2. An expression for the change in intensity on the axis of a lens and in the neighborhood of the principal focus.

3. Numerical results in a special case (ordinary crown and dense flint glasses) for the luminous efficiency of a lens of any dimensions, in so far as this efficiency is controlled by the chromatic aberration.

4. A comparison of the Chromatic and Spherical Aberrations in a special case, which shows that for the ordinary glasses the chromatic aberration and the color variation of the spherical aberration are of the same order of magnitude.

5. The probability that an appreciable improvement in the spherical correction would result from the use of a wave length in the neighborhood of 550mμ as a standard.

I. THE PROBLEM OF THE SECONDARY SPECTRUM

The focal length of a simple lens is given by the relation

$$1/f = (n - 1)\,(1/r + 1/r') \qquad (1)$$

and from this relation it is, of course, obvious that the focal length of a lens changes with any change in the index of refraction (n) of the glass. In particular, with a given lens the focal length changes with a change in the wave length of the incident light, and this change in focal length is the so-called longitudinal chromatic aberration of the lens.

This change in the focal length with change in wave length may be very much decreased by the use of a combination of two lenses, the chromatic aberrations of the one lens tending to counterbalance those of the other lens, the focal length of the combination being made the same for two, selected wave lengths. However, this compensation is not complete and there is always left a residual variation in focal length the "secondary spectrum" of an achromatic combination. This secondary spectrum may not affect injuriously the image in the case of lenses of short focal length and of small aperture, though it is troublesome when such lenses are used in a spectrometer, either of the ordinary or of the auto-collimating type. In the case of lenses of moderate focal length and of moderate aperture the effect of the secondary spectrum may be appreciable, and in the case of large, long-focus lenses of the

type used for astronomical telescopes, the injury to the definition and to the brightness of the image may be very serious.

The condition that a doublet shall have the same focal length for two given colors can be expressed in terms of the radii of the lenses and of the change in the index of refraction of the glass between the two colors. For a combination of two simple lenses in contact, the focal length may be written

$$1/\mathcal{F} = 1/f + 1/F \qquad (2)$$
$$= (n-1)(1/\rho) + (N-1)(1/\mathrm{P})$$

where the small letters denote quantities characteristic of one of the lenses, and the capital letters quantities characteristic of the other lens. As a compact notation

$$1/\rho = 1/r + 1/r'$$

and similarly for 1/P. The change in the focal length of the combination may be expressed as

$$\delta(1/\mathcal{F}) = \delta n/\rho + \delta N/\mathrm{P} \qquad (3)$$

and the condition, that the focal length shall not change with a change in wave length, is that the right hand side of this equation shall be zero. This may be expressed otherwise as

$$\frac{\delta N}{\delta n} = -\frac{\mathrm{P}}{\rho} \qquad (4)$$

The fulfilling of this condition is obviously possible in the case of any two wave lengths, for the change in the indices of refraction between the two wave lengths is perfectly definite and the radii of the two lenses may be so chosen as to make the ratio of P to ρ the same as that of $-\delta N$ to δn. However, when this is done, it is found, in general, that the focal length of the combination for other wave lengths is different from the focal length for the two chosen. The variations in focal length will be small, perhaps a few hundred-thousandths of the focal length for small changes in wave length, but if the range of wave lengths is at all broad the difference becomes more appreciable.

As an example, the measurements made by Keeler on the large refracting telescope of the Lick Observatory are tabulated below.[3]

Chromatic Aberration of the Lick Observatory 36″ Refractor

Spectrum Line	B	C	D	Minimum	F	G′	h
Wave Length	687	656	589	?	486	434	410
Increment in focal length, inches	0.00	−0.24	−0.45	−0.47	0.00	+1.45	+2.76
$\left(\frac{\delta\mathcal{F}}{\mathcal{F}}\right)$		−.00035	66	69		+.00212	404

[3] Keeler: Publications of the Astron. Soc. of the Pacific, Nov., 1890.

These figures are characteristic, and show clearly the manner in which the focal length increases from a minimum in the central part of the visible spectrum, the increment reaching the quite considerable figure of over three inches within the limits of the visible spectrum. Even with the relatively small aperture ratio of the Lick Telescope, about one to nineteen, one would certainly expect, as is found to be the case, that there would result a distinct injury to the sharpness and brightness of the image.

The indices of refraction for the glasses used in the Lick objective are not at hand, but I have these values for another celebrated objective, Fraunhofer's heliometer objective at the Koenigsberg Observatory. These are shown in Table 1 and, along with the indices, are tabulated the increments in index and the ratio of these increments, using the B line as the reference line in each case. It is obvious that this ratio is not constant. The ratio used by Fraunhofer for $\frac{P}{\rho}$ was very nearly the one tabulated under F.[4]

TABLE 1.

Spectrum Line	Wave Length	Indices of Refraction		Increment in Index		Ratio of Increments $\left(\frac{\delta N}{\delta n}\right)$
		Crown (n)	Flint (N)	δn	δN	
B	687	1.523 746	1.628 463	0	0	
C	656	1.524 738	1.630 307	0 992	1 844	1.860
D	589	1.527 357	1.635 451	3 611	6 988	1.936
E	527	1.530 726	1.642 271	6 980	13 808	1.980
F	486	1.533 699	1.648 455	9 953	19 992	2.008
G	431	1.539 271	1.660 623	15 525	32 160	2.066
H	397	1.543 985	1.671 168	20 239	42 705	2.104

1. CALCULATION OF THE CHANGE IN FOCAL LENGTH

The solution of the problem of the change in focal length with wave length is contained in equation (3). This equation, however, involves the index of refraction directly and in order to get a convenient solution of the problem it is desirable to express the indices in terms of wave lengths, and to express this dependence in such a form that it will be possible to calculate the change in focal length corresponding to any desired and small change in wave length. For such a purpose direct

[4] v. Merz, Zts. Instk *18*, p.288; 1898 (Referat)
Ste nheil, Zts. Instk. *19*, p.177; 1899.

experimental measurements of the indices are unsatisfactory and some sort of interpolation formula is desirable. A Cauchy formula of the type

$$n = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4} + \frac{d}{\lambda^6}$$

has been used by Czapski[5] and by H. Dennis Taylor,[6] and Hartmann's interpolation formula has been employed by Harting,[7] and by J. Wilsing.[8] Hartmann's formula seems to fit the experimental observations over a rather wider range than is needed for the present purpose, and as it is in a convenient form, it has been used as a basis for all calculations. The results obtained are good, therefore, only in so far as this formula adequately represents the variations in the indices as a function of the wave length.

Harting has derived an expression, with the aid of Hartmann's formula, for the change in focal length, but it has seemed to me desirable to put the expression in another form. The expression which I have used is in terms of the wave length of minimum focal length, and with it the calculations of the change in focal length with change in wave length can be made easily and quickly.

In order to obtain an expression for the proportional variation in focal length $\left(\frac{\delta.\mathscr{F}}{\mathscr{F}}\right)$ it is necessary (1) to express the variations in the indices in equation (3) in terms of the wave length and of the constants of Hartmann's formula, which are characteristic of the glasses, and (2) to obtain in terms of these same constants and of $\mathscr{F}$ values of ρ and P which will bring the minimum focal length to the desired value.

Hartmann's formula is[9]

$$n = n_0 + \frac{c}{(\lambda - \lambda_0)^a} \qquad (5)$$

in which the exponent "a" is not usually much different from unity. In some types of glass it appears that a sufficient accuracy can be obtained by using the value $a = 1$. In particular, this is true of some of the types of glass commonly used for telescope objectives, and consequently this constant is assumed to be unity in this paper. For the other quantities in (5) small letters will be used to designate the crown glass and the corresponding capital letters to designate the flint glass.

[5] Czapski, Zts. Instk., *6*, p. 342; 1888.
[6] Taylor, Monthly Notices Royal Astr. Soc., *54*, p. 67; 1894.
[7] Harting, Zts. Instk., *31*, p. 72; 1911.
[8] Wilsing, Zts. Instk., *26*, p. 41, 1906.
[9] Zs. Instk., *19*, p. 57, 1899. (Referat.)
Zs. Instk., *37*, p. 166; 1917.

The expression for the increment in index in changing from a wave length λ_1, the assumed wave length for minimum focal length, to any other wave length, λ, is, for the crown glass,

$$\delta n = \frac{(\lambda_1-\lambda)c}{(\lambda-\lambda_0)\ (\lambda_1-\lambda_0)} \tag{6}$$

and

$$\delta N = \frac{(\lambda_1-\lambda)C}{(\lambda-\Lambda_0)\ (\lambda_1-\Lambda_0)}$$

for the flint glass. For infinitesimal differences these expressions reduce to

$$\frac{dn}{d\lambda} = -\frac{c}{(\lambda_1-\lambda_0)^2} \qquad \frac{dN}{d\lambda} = -\frac{C}{(\lambda_1-\Lambda_0)^2} \tag{7}$$

The condition that the focal length of the combination be a minimum at the wave length λ_1 is that $\left(\frac{d\mathscr{F}}{d\lambda}\right)$ be zero for that wave length, or, differentiating the quantities in equation (2),

$$-\frac{1}{\mathscr{F}^2}\cdot\frac{d\mathscr{F}}{d\lambda} = \frac{1}{\rho}\frac{dn}{d\lambda} + \frac{1}{\mathrm{P}}\frac{dN}{d\lambda} = 0 \tag{8}$$

Now substituting for the derivatives the expressions of equations (7), we have one relation between P and ρ, which may be written

$$\frac{c}{\rho}\frac{1}{(\lambda_1-\lambda_0)^2} = -\frac{C}{\mathrm{P}}\frac{1}{(\lambda_1-\Lambda_0)^2} \tag{8a}$$

The values of P and ρ can now be obtained by combining (8a), and (2), and these are

$$\frac{1}{\rho} = \frac{\frac{1}{c}(\lambda_1-\lambda_0)^2}{\left(\frac{n_1-1}{c}\right)(\lambda_1-\lambda_0)^2 - \left(\frac{N_1-1}{C}\right)(\lambda_1-\Lambda_0)^2} \times \frac{1}{\mathscr{F}}$$

$$\frac{1}{\mathrm{P}} = \frac{\frac{1}{c}(\lambda_1-\Lambda_0)^2}{\left(\frac{N_1-1}{C}\right)(\lambda_1-\Lambda_0)^2 - \left(\frac{n_1-1}{c}\right)(\lambda_1-\lambda_0)^2} \times \frac{1}{\mathscr{F}} \tag{9}$$

If now the values of δn, δN, ρ, and P from (6) and (9) be substituted in equation (3), and if the assumption be made that it is permissible to write at the last

$$\delta\frac{1}{\mathscr{F}} = -\frac{1}{\mathscr{F}^2}\cdot\delta\mathscr{F}$$

there results the following expression for $\frac{\delta \mathcal{F}}{\mathcal{F}}$

$$\frac{\delta \mathcal{F}}{\mathcal{F}} = \frac{(\Lambda_0 - \lambda_0)}{\frac{n_1 - 1}{c}(\lambda_1 - \lambda_0)^2 - \frac{N_1 - 1}{C}(\lambda_1 - \Lambda_0)^2} \times \frac{(\lambda - \lambda_1)^2}{(\lambda - \lambda_0)(\lambda - \Lambda_0)} \quad (10)$$

$$= K_1 \times \frac{(\lambda_1 - \lambda)^2}{(\lambda - \lambda_0)(\lambda - \Lambda_0)}$$

In this expression for $\frac{\delta \mathcal{F}}{\mathcal{F}}$ the factor K_1 may be treated as a constant for any doublet, inasmuch as this constant involves only the constants of the glasses of the doublet and the value of the wave length, λ_1. It does not invo've the variable wave length, λ.

That the shape of the curve, which represents $\frac{\delta \mathcal{F}}{\mathcal{F}}$ as a function of λ, will depend upon the value of the wave length λ_1 may be seen from an inspection of formula (10), or from the curves of Fig. 1.

2. SOME NUMERICAL VALUES

As an example of the application of equation (10) I have calculated the proportional change in focal length for a doublet made from an "ordinary crown" (Schott's 0-60) and a "dense flint" (Schott's 0-103) glass. The constants of these glasses have been calculated by Harting[10] and by Wilsing.[11] The two sets of constants are slightly different but it would make no essential difference if one set were used instead of the other. The values are:

	Crown Glass (0-60)			Flint Glass (0-103)		
	n_0	$\log c$	λ_0	N_0	$\log C$	Λ_0
Harting:	1.49948	0.8854	172.3	1.58750	1.0944	209.4
Wilsing:	1.49942	0.88791	171.4	1.58809	1.08040	213.7

I have used Wilsing's values, except for the quantity $(\Lambda_0 - \lambda_0)$, for which I have used 40, a value intermediate between the two sets.

The values of K_1, equation (10), for various values of λ_1 are:

λ_1	500	520	540	550	560	570	580	600
K_1	.0132	.0120	.0111	.0105	.0101	.0097	.0093	.0086

The results of this calculation are represented graphically in Fig. 1 for three values of λ_1, 500, 560, and 600mμ. In addition there is shown

[10] Harting, Zs. Instk., *28*, p. 273; 1908.
[11] Wilsing: Zs. Instk., *26*, p. 41; 1906.

on Fig. 1 Keeler's measurements of the change in focal length for the Lick objective.

From curves similar to these I have calculated by graphical interpolation, for various values of λ_1 the wave length on the long wave length

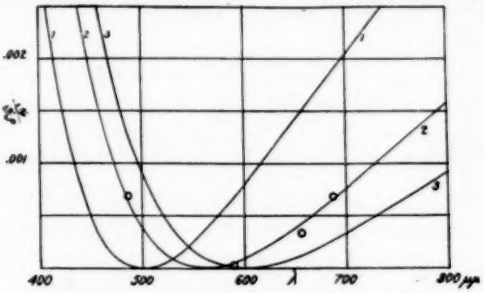

FIG. 1. *Change in Focal Length with Wave Length* (λ)
Curve 1 $\lambda_1 = 500m\mu$
2 $\lambda_1 = 560m\mu$
3 $\lambda_1 = 600m\mu$
Glasses: Schott 0.60, crown, and 0-103, flint.
The four circles indicate Keeler's measurements on the Lick objective at the B, C, D, and h lines. His values at G'(434) and h(410) would be very close to the calculated values for the continuation of curve 2.

side of λ_1 at which the focal length is equal to that for the *F* line (486mμ), and the results of this calculation are shown in Fig. 2. This curve, then, shows the wave length which must be used together

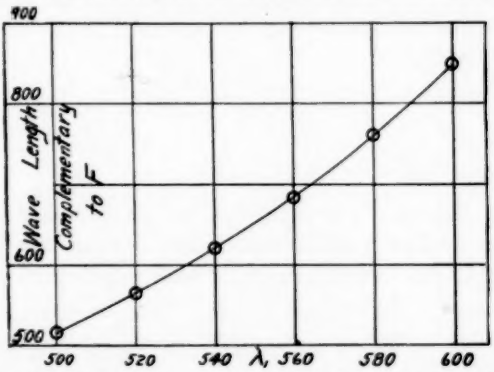

FIG. 2. *The relation between the wave length of minimum focal length* (λ_1) *and the wave length for which the focal length becomes equal to that for the F line. Glasses: Schott 0-60 and 0-103.*

with 486mμ, in order to obtain a given type of achromatization. It may serve, of course, for the converse problem, and from it one deduces

that when the *C* and *F* lines are made to focus at the same point, the minimum focal length comes at a trifle over 550mμ, and that when the *C* and *B* lines are brought together, the minimum comes at a trifle over 560mμ, perhaps 561. With the smaller telescopes, apparently the former type of achromatization is met with; with the larger telescopes, it is more nearly the latter. The choice is, probably, an empirical one, but the second, at least, of the two seems to be justified by the further calculations of this report.

3. ELIMINATION OF THE SECONDARY SPECTRUM

The two successful attempts to diminish the variation in focal length with wave length are dependent upon:

1. The use of the "new" optical glasses. In this case it appears that the aperture cannot be made large on account of the difficulty in correcting for the spherical aberration.[12]

2. The use of three types of glass by H. Dennis Taylor.[13] This combination consists of two converging lenses, one flint and one crown, and one diverging lens of flint, the three types being so chosen and the lenses so proportioned that the mean of the dispersions for the collective lenses very nearly balances the dispersion of the dispersive lens for a wide range of wave lengths. By this construction the variation in focal length is reduced to less than one-twentieth of that obtained with the ordinary glasses, without any corresponding decrease in the aperture of the objectives. One wonders why this construction is not used in the objectives of spectrometers, where the additional cost for the small lenses involved would not be a very important item.

II. THE SECONDARY SPECTRUM AND THE BRIGHTNESS OF THE IMAGE

There have been two attempts made to estimate the damage to the image resulting from the variation in focal length with wave length.

The first was made by H. Dennis Taylor,[13] and his experimental attack upon the problem was avowedly an attempt to get a rough estimate of the damage done to seeing. He tried to determine experimentally the depth of focus of objectives under various conditions, and used as his criterion the distance out of focus which he would have to move an eye piece in order to cause the diameter of the diffraction image of a star to double in size. Then, if for a given color the change

[12] Wolf: Zts. Instk., *19*, p. 1; 1899.

[13] H. Dennis Taylor: Monthly Notices, Royal Astr. Soc., p. 309, March, 1894, System of Applied Optics.

[13] Loc. cit.

in focal length from the reference color is greater than this distance, he considers that the given color does not contribute to the image. If the change is less, he counts the contribution as though the change in focal length were zero. This amounts to assuming that the image is produced by a band of wave lengths in the middle of the spectrum, the focus being perfect in this range, and that all of the light outside of this band is entirely eliminated. The visual effect of the various wave lengths within this band is weighted according to Abney's visibility data. In this way he obtained a visual efficiency of about 73 per cent for the Lick telescope, by graphical interpolation from Keeler's measurements of $\frac{\delta \mathcal{F}}{\mathcal{F}}$. In other words, the image obtained with the Lick telescope would be only 73 per cent as intense as one should obtain from an otherwise similar telescope in which the secondary spectrum was entirely wanting. He considered that 73 per cent was probably too high, and the present work would indicate that in this estimate he was correct.

Strehl was incited by Taylor's work to make a further investigation which Strehl[14] based upon Lommel's theory of the diffraction pattern due to a circular aperture. His calculations gave him, in terms of Keeler's experimental color curve for the Lick Observatory telescope, the ratio of the intensity of the light on the axis with the Lick Objective to the intensity which would have been obtained from an otherwise similar objective in which the secondary spectrum was entirely wanting. His work differed from Taylor's in that Strehl estimated the relative intensity at the center of the diffraction pattern at small distances from the true focus, measured along the axis of the telescope, in terms of the intensity at the focus. Multiplying this fraction by the relative sensitivity of the eye at the different wave lengths, and summing the results throughout the effective range of wave lengths, he obtained a measure of the visual efficiency.

Strehl's method of attack seems to me to be fundamentally sound and I have used it below. Not having access, however, to his interpretation of Lommel's work, I have worked out an expression for the variation of the intensity along the axis in terms of $\delta \mathcal{F}$ and the constants of the lens, and the results of calculation with this formula agree at least in order of magnitude with the results given by Strehl for a given $\frac{\delta \mathcal{F}}{\mathcal{F}}$.

[14] Zts. Instk., *17*, p. 50, p. 77; 1897.

1. AN EXPRESSION FOR THE VARIATION IN INTENSITY ALONG THE AXIS

The quantity which is wanted is I/I_0, where I represents the energy flow per second across a small area perpendicular to the axis and I_0 is the energy flow across a small area at the true focus for light of the wave length under consideration. This latter quantity can be easily expressed in terms of the light which comes up to the lens. A sufficient expression for our purpose is that I_0 is proportional to S^2, where S represents the effective area of the objective, an expression which assumes (1) that the spherical aberration for this wave length is nil, and (2) that the loss of light by reflection, absorption, and so forth is the same per unit area of the incoming beam, no matter what part of the objective is considered. This latter should not be far from the truth.

In order to obtain the intensity at a point away from the principal focus, it is necessary to calculate the amplitude at the point in question, and then to assume that the intensity is proportional to the square of the amplitude.

In calculating the amplitude the assumption is made that the resultant displacement at a point may be obtained by adding together the elementary displacements which would be produced at that instant by the light coming from the various elementary surfaces into which the incoming wave may be divided for the purpose of analysis. The maximum resultant displacement with respect to time is, of course, the desired amplitude. In the present case the conditions are relatively simple. A portion of a spherical wave, symmetrical about an axis (OP of Fig. 3) is converging to a point F, and it is desired to calculate the amplitude at a point P, on the axis and not very far from F.

In order to sum the elementary displacements it is necessary to weight the contributions made by the different elementary areas on the advancing wave. The assumption here made is the one referred to in the first paragraph of this section, namely, that equal areas on the advancing wave front, *before refracton at the lens*, contribute equally to the resultant vibration. It is the assumption commonly made when it is assumed that the light flux contributing to the formation of an image is proportional to the square of the linear aperture of the lens.

For the amplitude one has, on this basis,

$$dA = k \cdot dS \cdot \cos(\theta - \theta_0) \qquad (11)$$

where A represents the amplitude

k is an arbitrary constant

dS is an element of area and is measured on the incident wave before refraction at the lens

θ is a phase angle, which is zero for the light which travels along the axis

θ_0 is an arbitrary phase angle, the value of which must be so chosen as to make the integral of the quantity on the right hand side of equation (11) a maximum, for only in such a case will the integral give the resulting amplitude.

It is possible to express dS in terms of θ and in turn to express θ in terms of δ and the constants of the lens. Doing the latter first, one gets

$$\theta = 2\pi\left(\frac{\delta}{\lambda}\right) = \frac{2\pi}{\lambda}(PQ-PO) = \frac{2\pi}{\lambda}\cdot PF\,(1-\cos\phi) \qquad (12)$$

$$= \frac{\pi}{\lambda}\cdot PF\cdot\frac{h^2}{FQ^2}$$

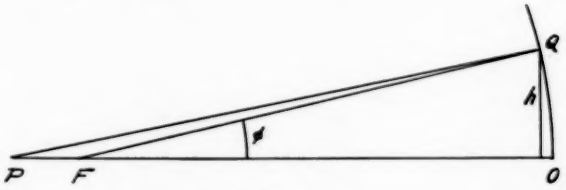

FIG. 3

where the additional symbols introduced should be clear from an inspection of Fig. 3. δ is, of course, the path difference which gives rise to the phase difference θ.

To express dS in terms of θ, one puts

$$dS = 2\pi h\cdot dh = \frac{\lambda\cdot\overline{FQ}^2}{PF}\cdot d\theta \qquad (13)$$

where the relation between dh and $d\theta$ is obtained from (12). Substituting in (11) and integrating between the limits of $\theta=0$ and $\theta=2\alpha$, the result obtained is

$$A = k\cdot\frac{\lambda\cdot FQ^2}{FP}\left[\sin(2\alpha-\theta_0)+\sin\theta_0\right] \qquad (14)$$

and this expression has a maximum value when θ_0 is equal to α, so that we have, finally, for A

$$A = k\pi h^2\frac{\sin\alpha}{\alpha} \qquad (15)$$

which should give, for $\alpha=0$, the value of $k\cdot\pi h^2$, as it does.

From this last expression, one gets immediately

$$\frac{I}{I_0} = \left(\frac{A}{A_0}\right)^2 = \left(\frac{\sin a}{a}\right)^2 \tag{16}$$

where a can be written from (12) as

$$a = \frac{\pi}{2} \cdot \frac{\delta \mathscr{F}}{\lambda} \cdot \left(\frac{H}{\mathscr{F}}\right)^2 \tag{17}$$

for a is by definition the half of the maximum value of the phase angle θ, PF is obviously equal to the $d\mathscr{F}$ which we have been using, H indicates the maximum value of the variable distance h, and $FQ = OF$ is the focal length of the lens combination ($\mathscr{F}$).

(16) and (17) contain the solution of the problem for monochromatic light. With white light the sum of a number of such terms, will be needed.

2. THE CALCULATION OF THE INTENSITY ON THE AXIS

By combining the results of the first section (I) with the results just obtained, one can get a measure of the intensity of the image which would be formed by light for which the energy distribution throughout the spectrum is uniform with respect to wave length and this will give a measure of the brightness of the image to an eye which is uniformly sensitive for the range of wave lengths considered. Inasmuch as these conditions are not in general satisfied, the results which we are about to obtain will give neither the visual brightness nor the intensity in any actual case. What it will give will be a measure of the behavior of the telescope, which can be used to get these quantities.

It seems desirable here to turn to numerical examples. If one substitutes in (16) from (17) and from (10), one gets an expression for the intensity in terms of the wave length, but the expression is clumsy and, I think, not very informing. As an example, I shall follow Taylor and Strehl and use a lens of the dimensions of the Lick Observatory objective assuming that the color curves of the objective are similar to the ones given in Fig. 1. It will be found possible to extend these results so as to give information with reference to a lens combination of any desired dimensions.

I shall follow Strehl further in not paying any attention in the calculation to values of a greater than π, or, in other words, I shall neglect the contribution to the intensity on the axis of all light for which the path difference from the center and from the edge of the lens is greater

than one wave length. The value of I/I_0 in all such cases would be small, and for visual purposes the luminosity of such wave lengths would be likewise small, so that the neglect of such terms is, it seems to me, justifiable, at least when one is considering the visual brightness. If it were desired to add these additional terms, there would be no great labor involved in calculating them.

One set of calculations I shall give in detail (Table 2), and it will probably be sufficient to give in graphical form the results of other similar calculations.

TABLE 2.

Numerical Example of Calculations
Pseudo-Lick Objective
$\lambda_1 = 560m\mu$

Dimensions and Constants of the Lens

Focal Length 57 ft. *Linear Aperture $2H = 36$ in.*

$$\frac{H}{\mathcal{F}} = \frac{1}{38} \qquad a = 1.89 \cdot \frac{1}{\lambda} \cdot \frac{\delta\mathcal{F}}{\mathcal{F}}$$

I λ	II $\frac{\delta\mathcal{F}}{\mathcal{F}}$	III a	IV $\frac{I}{I_0}$	V V	VI $V \cdot \frac{I}{I_0}$	VII E_λ (6000°)	VIII $E\lambda \cdot V \cdot \frac{I}{I_1}$
525	.0001118	3.66	.00	.75	.00	.80	.00
530	799	2.85	.01	.84	.01	.87	.01
535	538	1.90	.25	.90	.23	.93	.21
540	335	1.17	.62	.96	.60	.98	.59
545	182	.63	.87	.98	.85	1.00	.85
550	079	.27	.98	.99	.97	1.00	.97
555	019	.06	1.00	1.00	1.00	.99	.99
560	000	.00	1.00	.99	.99	.98	.97
565	018	.06	1.00	.98	.98	.96	.94
570	071	.24	.98	.96	.94	.93	.87
575	156	.51	.89	.92	.82	.89	.73
580	270	.88	.76	.87	.66	.84	.57
585	410	1.32	.54	.81	.44	.79	.35
590	576	1.84	.27	.75	.20	.72	.14
595	764	2.43	.07	.70	.05	.66	.03
600	974	3.07	.00	.64	.00	.60	.00
			924		874		822

The figures in columns I to IV of Table 2 are the ones with which we are at present concerned. The other columns take into account the

visibility of the eye and the distribution of energy in the spectrum of the source.

I have made calculations similar to those indicated in column IV on a number of different assumptions, and some of the results are indicated in Fig. 4. Fig. 4 shows three curves, representing three sets of values for I/I_0 as a function of the wave length. The three sets are calculated for different values of the wave length, λ_1, of minimum focal length, and for more ready comparison of the curves the values of I/I_0 are represented as a function of $(\lambda-\lambda_1)$ instead of as a function of λ. The curves are naturally somewhat similar, but there is a gradual broadening of the curves as the peak is changed in the direction of the longer

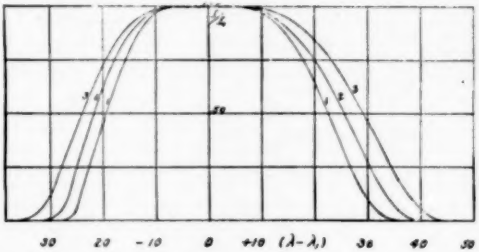

FIG. 4. *The relation between the wave length and the intensity at the λ_1-focus for a telescope objective of the size of the Lick refractor.*

Curve 1 $\lambda_1=540m\mu$
2 $\lambda_1=560m\mu$
3 $\lambda_1=580m\mu$

wave lengths. The area under the curves furnishes a measure of what might be called the intrinsic intensity which the telescope would be capable of providing, and measured in this way the goodness of the telescope would, of course, increase as the position of minimum focal length is moved toward longer wave lengths. The trend of this increase is shown in curve 1 of Fig. 5.

The curves of Fig. 4 are significant only in so far as they provide a step toward the solution of our problem. In this connection it should be noted that these curves would be modified if the intensity is calculated at a point on the axis other than the λ_1—focus, and in particular, the curves would be broadened and the area under the curves increased if the point chosen should be a trifle outside of this particular focal point.

III. THE EFFECT OF THE VARIABLE SENSITIVITY OF THE EYE

Beginning with Ives's papers in the Philosophical Magazine in 1912 there has been in the last dozen years almost a flood of papers dealing

with the relative visibility of the various wave lengths of the visible spectrum. Data have been accumulated for hundreds of subjects, and in a recent paper by Priest[15] these data have been graphically summarized.

These curves all have a general trend which is much the same, and from them it is possible to pick out a set of values for the relative visibility which should not much misrepresent the variation in the

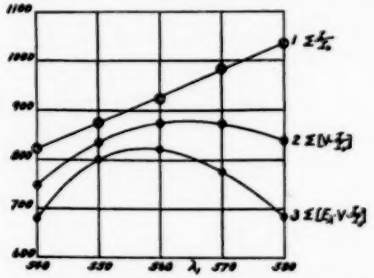

FIG. 5. *The total intensity at the λ_1-focus as a function of λ_1.*

Curve 1 $\Sigma \frac{I}{I_0}$ *Emitted energy the same at all wave lengths; Eye equally sensitive at all wave lengths*

2 $\Sigma\left[V \cdot \frac{I}{I_0}\right]$ *Emitted energy the same at all wave lengths; Visibility (V) taken into consideration*

3 $\Sigma\left[E_\lambda \cdot V \cdot \frac{I}{I_0}\right]$ *Source at 6000°K; Visibility taken into consideration*

Lens of the size of the Lick objective.

sensitivity of the eye with wave length in the case of persons who do not show marked defects in color perception. If one takes a set of such values (column 5 of table 2), and multiplies each one of them by a value similar to the ones which are tabulated in column IV of table 2, one gets a measure of the effect which will be produced upon the eye by the various wave lengths which enter into the formation of a star image. Column VI gives the product of the numbers in columns IV and V, and the numbers in column VI provide a measure of the effect produced upon the eye. The sums of the numbers listed in column VI gives, it seems to me, a significant measure of the excellence of the telescope, insofar as the color correction is concerned.

In Fig. 6 are represented the results of such a calculation for our pseudo-Lick telescope, assuming the wave length for minimum focal

[15] Scientific Papers of the Bureau of Standards, No. 417, 1921, on "Spectral Distribution of Energy required to evoke the Gray Sensation."

length to be 560 mμ. The four curves there shown represent the visual brightness for a uniform spectrum for four different positions on the axis of the lens. Curve 1 refers to the focus for the wave length λ_1,

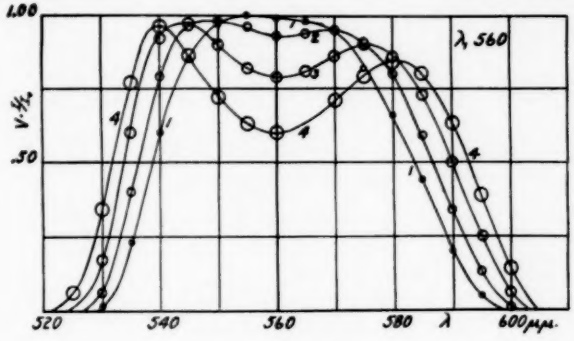

FIG. 6. *The relation between visual brightness and wave length* $(V \cdot \frac{I}{I_0})$

Curve 1 At the λ_1*-Focus,* $\lambda_1 = 560$ *mμ.*
2 0.2mm outside of the λ_1*-focus*
3 0.4mm outside of the λ_1*-focus*
4 0.6mm outside of the λ_1*-focus*

and curves 2, 3, and 4, respectively, refer to points which are about 0.2mm, 0.4mm, and 0.6mm further out on the axis. These three distances correspond to increments in the phase angle (α) of about

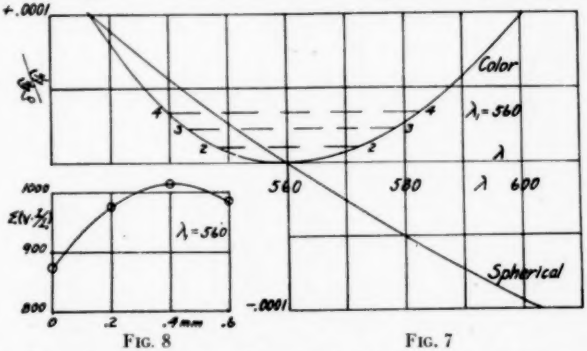

FIG. 8 FIG. 7

FIG. 7. *Comparison of the chromatic and spherical aberrations for a common type of objective of the aperture of the Lick refractor (F/19).*

45°, 90°, and 135°, or to a difference in the optical path between the center and the edge of the lens of about $\frac{1}{4}\lambda_1$, $\frac{1}{2}\lambda_1$, and $\frac{3}{4}\lambda_1$ respectively. The point corresponding to curve 3 gives probably as bright an

image as could be obtained under the conditions assumed for the lens and illumination. The areas under these curves are represented graphically in Fig. 8 and from this curve it appears that the brightness may be perceptibly increased by focussing on a point a little outside of the λ_1-focus.

For the same conditions for which the curves of Fig. 6 show the visual brightness, the "color" curve of Fig. 7 shows the corresponding values of $\frac{\delta \mathcal{F}}{\mathcal{F}}$. The value of $\frac{\delta \mathcal{F}}{\mathcal{F}}$ corresponding to curve 1 of Fig. 6 is, of course, zero for a wave length λ_1. The values of $\frac{\delta \mathcal{F}}{\mathcal{F}}$ for this same wave length and corresponding to the curves labelled 2, 3, and 4 in Fig. 6 are indicated by the similarly labelled broken lines across the bowl of the curve in Fig. 7. (3) is very close to the best focus for this particular lens.

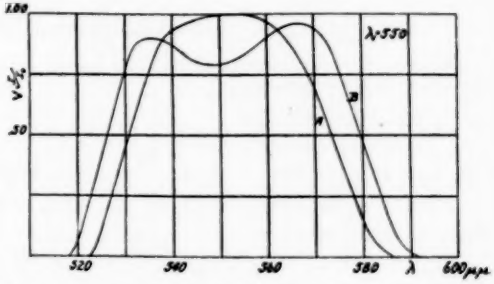

FIG. 9. *Curve A Visual brightness at the λ_1-focus, $\lambda_1 = 550m\mu$.*
B 0.4mm outside of the λ_1-focus
Lens of the size of the Lick objective
Compare with Fig. 6.

The curves in Fig. 9 are visual brightness curves, which differ from the ones just described only by having 550mμ instead of 560mμ as the wave length of minimum focal length, λ_1. Curve *A* represents the visual brightness at the λ_1-focus and curve *B* refers to a point which is about 0.4mm further out on the axis, curves *A* and *B* corresponding thus to curves 1 and 3 of Fig. 6.

By comparing the area under curves similar to those in Figs. 6 and 9 with the area of a relative visibility curve drawn to the same scale, there results an estimate of the efficiency of the objective under the assumed conditions. With the scale here used, the area under the visibility curve corresponds to 2028, and by division we get the following estimates of the efficiency with a uniform spectrum:

λ_1	540	550	560	570	580
Visual Efficiency					
λ_1 focus	.368	.411	.430	.432	.413
Best focus		.477	.500	.497	

The values here shown for the λ_1-focus correspond to the points shown on curve 2 of Fig. 5.

Insofar as these figures are significant, they indicate that the wave length corresponding to the minimum focal length should be about 560. This nearly corresponds with the practice of correcting large telescopes to give equal focal lengths for the *B* and *F* lines, as may be seen from Fig. 2.

IV. THE EFFECT OF THE ENERGY DISTRIBUTION IN THE SOURCE

Any estimate of the luminous efficiency of a telescope will have to take into account the energy distribution of the source as a function of wave length. This depends upon the effective temperature of the source, and this factor will be extremely variable, especially in the case of an astronomical telescope. However, some sort of an estimate of the effect of this factor upon the seeing is desirable, and I have chosen to calculate in terms of a black body source at a temperature of 6000°*K*. The energy distribution I have taken from Mr. Priest's convenient curves.[16] The numbers read from this curve are entered in column VII of Table 2 and the product of these numbers by the numbers entered in column VI gives the numbers entered in column VIII. These numbers represent, therefore, the effect upon the eye of a star at a temperature of 6000° when viewed with our pseudo-Lick telescope for a λ_1 value of 560mμ. The luminous efficiency here has a value of about 54 per cent, the number corresponding to the maximum possible intensity being in this case 1524. If the best focus is taken, the efficiency becomes about 61.2 per cent. These figures should correspond to nearly the performance of the Lick telescope.

If instead of making λ_1 equal to 560mμ, the telescope were achromatized for the *C* and *F* lines, making λ_1 equal to about 550mμ, the luminous efficiencies would be respectively 52.5 per cent and 59.4 per cent. If λ_1 be taken as 570, the values of these luminous efficiencies become 50.9 and 58.2 respectively. The results for the λ_1-focus are represented in curve 3 of Fig. 5.

[16] Loc. cit. p. 245.

It follows from these figures, I think, that under the assumed conditions the brightest image will be obtained for a λ_1 value of about 560mμ, or perhaps a little under, or that the telescope should be achromatized for F and B, or perhaps a little better for F and a wave length somewhat on the C side of B.

V. LUMINOUS EFFICIENCY FOR A LENS OF ANY DIMENSIONS

It is possible to generalize the results obtained in sections III and IV, because of the fact that the value of I/I_0 for a given wave length, depends upon the values of the parameter a only, and this phase angle can be expressed in terms of the focal length and the aperture of the lens. We have from equation (17) that

$$a = \frac{\pi}{2} \cdot \frac{1}{\lambda} \cdot \frac{\delta \mathcal{F}}{\mathcal{F}} \cdot \frac{H^2}{\mathcal{F}} \qquad (17')$$

In this expression the factor $H^2/\mathcal{F}$ is the only one which depends upon the dimensions of the telescope, for, of course, $\frac{\delta \mathcal{F}}{\mathcal{F}}$ depends only upon the types of glass which are used and upon the value chosen for λ_1. I have worked out for the glasses used above and for values of λ_1 of 550 and 560mμ the luminous efficiency as a function of the quantity $H^2/\mathcal{F}$ (dimensions in centimetrs), and the results of the calculation are represented in Figs. 10 and 11.

In Fig. 10 are collected the results of the calculations of the visual efficiency for a source of uniform spectral intensity, and in Fig. 11 are the corresponding values for the visual efficiency using a source at a temperature of 6000°K. The larger circles correspond to a λ_1 value of 560, the smaller to a λ_1 value of 550mμ. The single curve of Fig. 10 corresponds to the λ_1-focus. The three curves of Fig. 11 are drawn for different focal settings, the three corresponding to the three similarly labelled in Fig. 6. That is, curve 1 corresponds to the λ_1-focus, and curves 2 and 3 to points further out on the axis, for which the path difference for paraxial and peripheral rays are respectively $\lambda_1/4$ and $\lambda_1/2$. All of the curves are drawn for a λ_1 value of 560mμ, and the points for a λ_1 value of 550mμ are indicated by small circles for the purpose of comparison with the 560 mμ values.

From these results one may conclude that so long as $H^2/\mathcal{F}$. is small and insofar as a choice may be made on the ground of "visual efficiency" it is not a matter of great importance as to whether the lens is achromatized for the C and F or for the B and F lines. For large values of $H^2/\mathcal{F}$ the latter seems preferable.

One may furthermore deduce from these results the dimensions which a lens may have and at the same time have any desired "visual efficiency," in the sense in which the term is here used. If one assumes that 90 per cent. is a sufficiently great value for the efficiency, then the

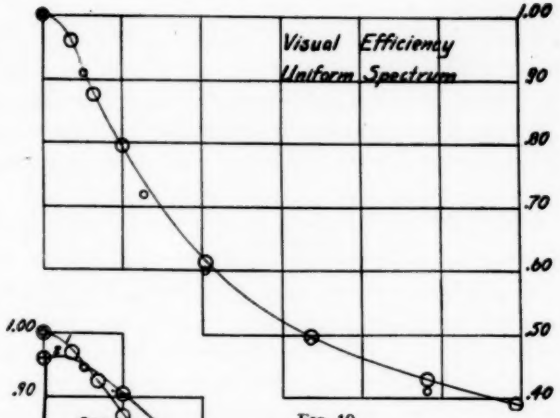

FIG. 10

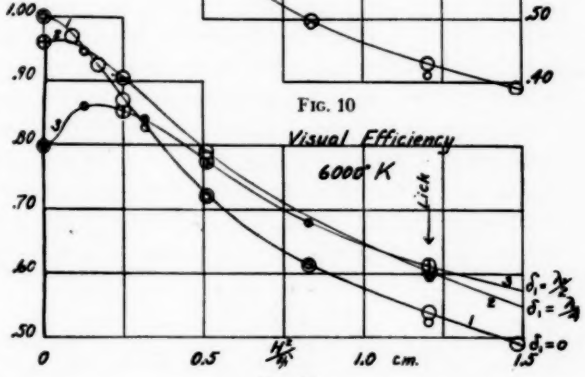

FIG. 11

FIG. 10. *Visual efficiency of an achromatic pair of any dimensions in terms of the parameter* $H^2/\mathcal{F}$ *cm. Uniform spectrum and* λ_1*-focus. Small circles,* $\lambda_1 = 550m\mu$*; large circles,* $\lambda_1 = 560m\mu$*. Glasses, ordinary crown and dense flint. The Lick objective is plotted at about 1.2 cm. The spherical aberration is assumed to be negligible.*

FIG. 11. *Visual efficiency for a source at 6000°K. Curve 1 is for the* λ_1*-focus, and curves 2 and 3 are for points outside the* λ_1*-focus, for which the path difference between paraxial and peripheral rays are respectively* $\frac{1}{4}\lambda_1$ *and* $\frac{1}{2}\lambda_1$*.*

curves of Fig. 11 show that at the best focus the maximum value of $H^2/\mathcal{F}$ is a little greater than one-fourth. This means that for a focal length of one meter, the lens may have a diameter ($2H$) of a little over ten centimeters. If one desired a similar performance with a telescope of the focal length of the Lick 36″ telescope, namely 1737 cm, then the lens could have the diameter

$$\frac{(2H)^2}{\mathscr{F}} = 1 \qquad \text{or} \qquad 2H = \sqrt{\mathscr{F}} = 41.7 \text{ cm}$$

For a lens of this same diameter, 36 inches, a ninety per cent performance may be expected for a focal length of $(91.5)^2$cm, or something over eighty meters.

From the point of view, then, of the color correction one should be permitted to predict that a telescope will give a satisfactory performance if the square of the lens diameter is not greater than the focal length of the lens, both measured in centimeters, and that the damage to the seeing will become marked if this ratio is much exceeded.

VI. THE EFFECT OF SPHERICAL ABERRATION UPON THE LUMINOUS EFFICIENCY

All of the calculation in the preceding sections has been on the assumption that the spherical aberration was in all cases and for all wave lengths negligible in amount. The second of these two assumptions needs to be investigated, and the results of an investigation indicate that when the spherical aberration is corrected for some one wave length the aberration for other wave lengths within the visible spectrum will not be negligible.

An expression for the change in the spherical aberration with change in index of refraction is given by H. Dennis Taylor in his "System of Applied Optics." The expression is, in terms of the Coddington notation,

$$d\left(\Delta\frac{1}{v}\right) = \frac{1}{8\rho^3}\left[\left(1-\frac{2}{n^2}\right)s^2 + 4\left(2n-1-\frac{1}{n^2}\right)s\cdot p + \left(9n^2-14n+3+\frac{2}{n^2}\right)p^2 + (3n^2-2n)\right]H^2\cdot dn \qquad (18)$$

where the symbols s and p are defined as follows

$$s = \frac{r'-r}{r'+r} \qquad p = \frac{v-u}{v+u}$$

On the assumption of two thin lenses in contact it is not difficult to calculate the value of $\delta\mathscr{F}/\mathscr{F}$ for a combination of two lenses which are achromatized in any desired way.

For the two types of glass in question, expression (18) reduces to the following numerical relations:

$$\delta\left(\Delta\frac{1}{v}\right)=\frac{1}{\rho^2}\ [0.0167\cdot s_1^2+0.803\cdot s_1\cdot p_1+0.820\cdot p_1^2+0.474]\cdot H^2\cdot\delta n$$
$$\delta\left(\Delta\frac{1}{v}\right)=\frac{1}{P^2}\ (0.0300\cdot s_2^2+0.933\cdot s_2\cdot p_2+0.593\cdot p_2^2+0.583)\cdot H^2\cdot\delta N \qquad (19)$$

If one further assumes a telescopic lens, then p_1 is -1, and p_2 has the value $\left(\frac{2F}{f}-1\right)$. If we choose to make the minimum focal length at 560mμ, p_2 will have the value 2.278. Choosing for the crown lens an equi-convex lens, $s_1=0$, and for the flint lens a plano-concave, $s_2=+1$, there results a combination which is nearly without spherical aberration, though with the ratio assumed for the rhos the two inner surfaces have slightly different curvatures. Substituting these values of s and p in equations (19), there result numerical values for the quantities in the parentheses.

$$\delta\left(\frac{1}{\mathcal{F}}\right)=\frac{H^2}{\rho^2}\ [.887\ \delta n-.765\ \delta N] \qquad (20)$$

ρ may be expressed in terms of the focal length of the objective and the values of δn and δN may be calculated with the aid of the Hartmann formula. Substituting the values so obtained in equation (20), values are obtained for $\delta\mathcal{F}/\mathcal{F}$ for various wave lengths in the neighboring of 560mμ. A set of such values are given in table 3. In this table under "Spherical aberration" the numbers given in the first line need to be multiplied by $\left(\frac{H}{\mathcal{F}}\right)^2$ in order to obtain the values of $\delta\mathcal{F}/\mathcal{F}$, and in the

TABLE 3.

Comparison of the Chromatic and Spherical Aberrations

Wave Length	500	520	540	560	580	600
$\frac{\delta\mathcal{F}}{\mathcal{F}}$	$\frac{H^2}{\mathcal{F}^2}\times 0.283$	Spherical +0.178	Aberration +0.080	0.000	−0.071	−0.133
$\left(\text{for }\frac{H}{\mathcal{F}}=\frac{1}{38}\right)$	+.000196	123	(+) 055	000	(−) 049	092
$\frac{\delta\mathcal{F}}{\mathcal{F}}$	.000385	Chromatic 151	Aberration (+) 034	000	(+) 027	097

second line the numbers given hold for the special value of $H/\mathcal{F}$ which characterizes the Lick objective. The numbers entered under "Chro-

matic aberration" are, in part, the same as those entered in column II of Table 2. The comparison of the two aberrations is made graphically in Fig. 7.

It appears from this that within the range of significant wave lengths the chromatic and the peripheral spherical aberration may be of the same order of magnitude, and that in general it is not permissible to neglect the latter in a consideration of the possible excellence of seeing which may be obtained with an objective.

VII. THE CHOICE OF WAVE LENGTHS IN LENS DESIGNING

The question as to what wave lengths it would be desirable to have specified in order to have the data most needed for computing optical systems is the one which was the occasion for this report. The answer to that question seems to me to be that it would be more desirable to have the constants of Hartmann's or some other similar formula than to have any table of wave lengths. This may not be practicable.

If indices are to be given directly, it seems to make not a great deal of difference, for most purposes, whether the lines chosen for achromatizing are the *F* and *C*, or the *F* and *B*, the latter being somewhat more desirable. For calculating the spherical aberration, the *D* lines do not in general seem to me to be a suitable choice. The choice of this wave length places the wave length of zero spherical aberration on the wrong side of the wave length of minimum focal length. The change in focal length due to chromatic aberration is always plus, whereas the change due to spherical aberration is plus on the short wave length and negative on the long wave length side of the wave length for which the combination is corrected. It would seem, therefore, that it would be better to have the spherical aberration corrected for a wave length a little shorter than the wave length of minimum focal length. The fact that the best focus will be obtained beyond this minimum has a bearing upon the question, and would make it desirable not to have the choice made at too great a distance on the short wave length side of λ_1- perhaps 2 to 4mμ. The mercury line at 546mμ might offer a suitable choice.[17]

Two things should be borne in mind in judging the general validity of these conclusions. In the first place, in all the calculations involving secondary spectra, the simpler form of Hartmann's interpolation formula is supposed to be adequate for such a calculation. In the second place, the numerical results all refer to a single pair of optical glasses,

[17] Cf. J. W. Gifford, Nature *114*, p. 645, 1924.

the "ordinary crown" and the "dense flint." If similar results should be desired for other combinations, it would be necessary to recalculate throughout. The work would, however, be niether difficult nor particularly tedious, inasmuch as with the exception of the constant K_1 in equation (10) the calculations can be made throughout with slide rule and graph.

The numerical results are further limited in that they apply only to to lenses which are to be used for visual purposes. If the doublet is to be used as part of a photographic combination, the characteristics of the photographic emulsion would need to be introduced into the analysis in place of the visibility data of the eye.

– – – – –

When this work was presented at the Cleveland meeting of the Optical Society, Mr. Rayton of the Bausch & Lomb Optical Company gave an example of the amount of chromatic aberration which could exist in a lens system without doing appreciable injury to the definition. In the case referred to, erecting prisms were introduced into a telescopic system and these prisms caused a difference in focus between the *C* and *F* lines of about .38mm. The telescope in question had a focal length of 120 cm and an aperture of 8 cm. On trial the image seemed equally good, with or without the prisms, a result which is in harmony with the present work.

The value of $H^2/\mathcal{F}$ for this lens is $4^2/120$ or .13cm, and from Fig. 11 it may be seen that for such a lens the intensity on the axis would be 95 per cent of that obtained with a perfectly achromatic lens, and that the difference between a *C*-*F* and a *B*-*F* correction is not visually appreciable. If this lens were corrected for *B* and *F*, then between *C* and *F* there would be a difference in focal length of about .31mm, which is of the same order as the observed .38mm introduced by the prisms. The prisms should, therefore, not be expected to materially injure the definition. It is gratifying to find that they do not.

BRACE LABORATORY,
UNIVERSITY OF NEBRASKA,
DECEMBER 29, 1923.

Contact Potential Differences between Glowed Metals in Vacuo.—Vieweg, working in Hallwachs' laboratory, undertook to measure the contact potential difference between strips of metal foil after each of a series of 15-second glowings because Hallwachs had observed that when the photoelectric current from the metal is measured in this manner it first rises and then sinks, and had inferred that

the effect of the consecutive glowings is first to remove a surface gas film and subsequently to expel occluded gas; if, therefore, the C.P.D. is altogether due to a surface gas film it might be expected to vanish with the first glowings, if to occluded gas then it might be expected to diminish gradually. Actually it was found in the observed cases (Al, Ni, Cu, Pt, Au) to change to some extent (but not to become zero) after the first glowing, and thereafter very little. Thus the evidence so far as it goes indicates that the C.P.D. is partly but not altogether due to surface gas film, and depends little or not at all upon occluded gas. The changes were all such as to make each metal more electropositive, and in the cases of Cu, Al and Ni were considerable portions of a volt, therefore of the order of the initial values. The method was the familiar one designated alternatively as Kelvin's or Pellat's; the details of the experiments are described at such interminable length as to make the article very tiresome reading. But the vacuum was not extremely good, 10^{-4} mm being the best claim made for it; only one or two samples of each metal were tested, and as the photoelectric current was not measured we have only analogy with Hallwachs' earlier results to indicate how far the de-gassing had progressed at each stage.—[R. Vieweg, Dresden; Ann. d. Phys. *74*, pp. 146-170; 1924.]

KARL K. DARROW

Formula for the Probability of Extraction of an Electron from an Atom by a Quantum.—The proposed formula is: probability $= A\lambda^3(n_K/\lambda^2{}_K + n_L/\lambda_L{}^2 + n_M/\lambda_M{}^2)$, in which λ represents the wave length of the quantum, λ_K, λ_L, λ_M the K, L, and M absorption wave lengths of the atom and n_K, n_L, n_M the number of electrons in the corresponding levels, and A is a constant. Any term for which the absorption wave length in the denominator is less than the irradiation wave length is dropped out. The different L-levels are considered as one, the different M-levels also as one, while the levels farther out, if any, are ignored for the present; the form of the expression suggests however the obvious generalization to take account of these complexities. The test of the formula as presented consists in comparing the values of the foregoing expression with the observed absorption coefficients for 7 elements ranging from number 4 to number 82, and for each of three wave lengths. The ratio of the observed absorption to the proposed expression in these 21 cases fluctuates between 1.21 and 2.18 while the absolute values of either vary between extremes standing in the ratio 1:9000. The comparison is made somewhat uncertain by the problem of how much to deduct from the apparent absorption coefficient to allow for the scattering. The agreement between the formula and the experimental data on the drop in total absorption occurring at the K-frequency is not so favorable.—[G. E. M. Jauncey, Washington Univ.; Phil. Mag. *48*, pp. 81-88; 1924.]

KARL K. DARROW

ON A METHOD OF CALCULATING THE RESISTANCE OF COILS AT VERY HIGH FREQUENCIES

By G. Breit

Introductory

The problem of skin effect at very high frequencies in a long single-layer solenoid has been discussed by A. Sommerfeld[1] and also by Lenz.[2]

The assumptions made in the above discussions restrict the applications of the results to the case of single-layer solenoids which are extremely long, wound with a very small pitch (in comparison with the diameter), and which are used at frequencies low enough to justify quasi-stationary considerations. The above two papers are concerned both with low and high frequencies. The present discussion is concerned only with high frequencies i.e. such frequencies that the thickness of the skin to which the current is confined is negligible in comparison with the thickness of the wires. The frequency however is not supposed to be so high as to make quasi-stationary considerations inapplicable.

For high frequencies Sommerfeld discusses only a special case, namely that of perfectly close winding of the coil and finds for this case that the resistance of the coil is higher in the ratio of 3.73 to 1 than the resistance of its wire when stretched out in a straight line. This result is contrary to a later result obtained by H. Abraham, L. Bloch, and E. Bloch,[3] who by an arithmetical method of successive approximations obtained for this ratio the value 3.46.

Lenz does not consider the case of perfectly close winding and assumes the result of Sommerfeld to be correct for this case. Thus, *there is a divergence of opinion as to the true value of the ratio of the resistance of a closely wound single-layer solenoid to the resistance of its wire at the same very high frequency when the wire is stretched out in a straight line.*

Further, the calculation of algebraic formulas for the resistance of infinitely long single-layer solenoids of sufficiently small pitch but not

[1] A. Sommerfeld, "Über den Wechselstromwiderstand von Spulen." Ann. d. Physik, *329*, p. 609, 1907.

[2] W. Lenz, "Über die Kapazität der Spulen und deren Widerstand und Selbstinduktion bei Wechselstrom." Ann. d. Physik, *342*, 1912, p. 923.

[3] See Sur le calcul des Pert..s dans les selfs d'antennes en tubes de cuivre. Note de H. Abraham, L. Bloch and E. Bloch, Radiotelegraphie Militaire, 1919, "E. C. M. R. Report No. 4629."

necessarily of perfectly close winding, has been undertaken only by Lenz and although the formulas of Lenz are capable of giving numerical results, they become somewhat involved when a high accuracy is desired and become slowly convergent when the condition of perfectly close winding is approached.

Finally all of the investigations, above mentioned, deal exclusively with infinitely long, single-layer solenoids.

It is intended in the present paper to supply some of the information, the lack of which has been just pointed out. Thus it is intended to treat the coil under as general conditions as are amenable to two dimensional considerations. It is found that the restriction made by assuming the solenoid to be infinitely long and to be wound in a single-layer can be dispensed with, provided one can assume instead that the number of turns in the coil is so large that the current distribution is constant in a portion of the coil which is large enough to contain an appreciable number of turns (e.g. 20), and which is small in comparison with the whole coil, and provided the separation between the layers of the coil is large in comparison with the pitch.

For this case the formulas for the resistance will be worked out and the results will be compared (in the special case of perfectly close winding) with those of A. Sommerfeld, and H. Abraham, L. Bloch and E. Bloch.

DERIVATION OF THE TWO-DIMENSIAL EQUIVALENT OF THE COIL SECTION

The assumptions made in the following treatment are:

(a) The pitch of the winding is small in comparison with the diameter of the coil.

(b) The number of turns in each layer of the coil is so large that the current distribution in a cross-section of the wire is the same for all wires in a portion of the coil which is large enough to contain an appreciable number of turns, (e.g. 20), but which is small in comparison with the whole layer, and in comparison with the diameter.

(c) The frequency is so low that a quasi-stationary treatment is applicable.

Assumption (b) makes it possible to apply two-dimensional considerations to a portion of a layer which is short in comparison with the length of the layer and with the diameter of the coil but which is long enough to contain an appreciable number of turns.

Since the diameter of the coil is large in comparison with the length of the portion considered, the curvature of the winding may be neglected in writing down the field equations for the region of space in the immediate neighborhood of the section considered.

If, however, one were to discuss the behavior of the whole coil the curvature of the winding could not be neglected in discussing the field equations.

However, one can form a judgment as to the nature of the magnetic field caused at the central wire of the section considered by the large part of the coil which is left after cutting out the small section and thus one can avoid using three dimensional calculations.

In fact (See Fig. 1) let $ABHG$ be the layer to which the section $ECDF$ belongs. Let the turn PP' be the central turn of that section, and let $PA = P'B = PA' = P'B'$.

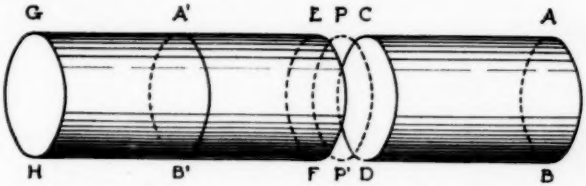

FIG. 1. *Illustration of division of coil.*

Consider the effect at PP' of the current in the section between AB and $A'B'$. The magnetic field caused at PP' by this current is by symmetry parallel to the axis of the solenoid.

Thus the component of magnetic intensity perpendicular to the axis of the solenoid is due to $A'B'HG$ and on account of the smallness of the diameter of the wire in comparison with EC varies slowly within the cross-section of the wire. Its value can be taken as that derived from current sheet considerations.

A similar condition exists for the component of magnetic intensity parallel to the axis of the solenoid.

For in this case turns situated far from the edges EF or CD do not give a rapidly varying magnetic field at PP' on account of their distance from PP', and turns close to the edges EF, CD give a magnetic field which is nearly uniform at PP' because in the discussion of this magnetic field the curvature of the winding could be neglected and these turns gives to a first approximation no field at P.

It thus may be supposed that *within the cross-section of the turn PP' the magnetic field superposed by the large portion of the coil is uniform.*

In addition to this field there is the field of the remaining turns in the section *ECDF*. This field may be computed from two-dimensional considerations as soon as the current in these turns is known. According to the assumption (b) this current is the same in amount and distribution as the current in the turn *PP'* itself.

Thus the problem has been reduced to the following.

A large number of straight infinitely long, equally spaced, equally thick, round wires is given. It is required to find the current distribution in the central wire, when it is known that the total current and the current distribution is the same in all the other wires and that in addition to the magnetic field of these other wires a uniform magnetic field is superposed on the central wire. (This is the field due to the large section of the coil, and has been computed in the previous paper already mentioned.)

The problem can be further idealized by taking the number of wires considered to be infinite and by agreeing that this infinity is to be regarded as the limiting case of a finite number as this number is increased indefinitely by adding equal numbers of turns on both sides of the central wire.

The solution of this problem will be synthetic.

Suppose that the uniform magnetic field superposed on the central wire is also superposed on all the others. Then it can equally well be imagined to be impressed on all of the space and to contribute to the magnetic intensity not only inside the wires but also everywhere around them.

Such a magnetic field has no singularities and thus could be imagined to exist in the two dimensional model sketched above without bringing into play any other currents but those in the infinite row of straight parallel wires.

Further, no matter what the current distribution in any particular wire may be, the magnetic field caused by that wire at a point sufficiently far from it is numerically equal to

$$\frac{2I}{10r}$$

where I is the current, and r is the perpendicular distance from the point on the wire, and this magnetic field is perpendicular to the distance r, its sense being given by the right handed screw law.

If then the magnetic field of the infinite row of wires is considered at some point far removed from the plane of axes of the wires, then this

field may be computed as if the current distribution in the wires were uniform.

The field is then nearly parallel to the plane of axes and directed in opposite senses on both sides of this plane. In addition to this field there is also the impressed field, which as was explained, is uniform everywhere.

Consequently at an infinite distance in a direction perpendicular to the plane of axes of the wires the field is uniform.

Further the magnetic intensity is intimately connected with the current distribution.

In fact, at high frequencies the current flows in a thin skin at the surface of the wire, and it may be shown that at a given distance from the surface of the conductor (if this distance is small) the current density has a value which is the same fraction of the current density at the nearest point of the surface independently of where this nearest point may be located on that surface. This current distribution in turn may be replaced so far as calculation of resistance is concerned, by the equivalent assumption of the current flowing in a thin skin whose thickness is the same for all points on the surface of the conductor. The justification of this is to be found in the work of Sommerfeld (l.c. p. 65) and H. Abraham, L. Bloch, and E. Bloch (l.c. p. 65). Consequently it remains to find what total current is carried by the whole thickness of the skin in a portion of it having unit length. In other words it is necessary to know the current per unit length of the surface of the wire. Thus it is necessary to know the total current flowing through the skin adjacent to the part of the contour at AB. This total current is proportional to the magnetic field at AB.

In fact the magnetic field vanishes inside the conductor. Therefore its line integral along a curve CD inside the conductor is zero. Since

$$curl\ \mathbf{H} = \frac{4\pi}{10}\mathbf{U}$$

the line integral of **H** around the contour $EFCD$ is equal to the flux of $\frac{4\pi}{10}\mathbf{U}$ through the area $EFCD$ i.e. to the total current through the area. But the line integral of H vanishes over CD. It also vanishes over FC and ED if these are normal to EF because then they are also normal to the field. Therefore, if H is the numerical value of **H**,

$$H \cdot \overline{EF}$$

is the line integral of **H**.

If u is the current density per unit length, the surface integral of $\frac{4\pi}{10}\mathbf{U}$ is

$$\frac{4\pi}{10} u \cdot \overline{EF}$$

As explained the surface integral and the line integral are equal. Hence

$$\frac{4\pi}{10} u \cdot \overline{EF} = H \cdot \overline{EF}$$

or

$$u = \frac{10}{4\pi} H$$

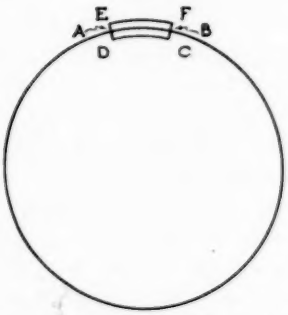

FIG. 2. *Diagram for discussion of relation of current density and magnetic field*

The problem of finding u is therefore reduced to the problem of finding H.

The conditions for **H** imposed by the conditions of the problem are:

(a) $$\text{div } \mathbf{H} = 0.$$

everywhere outside the wires.

(b) Everywhere outside the wires the line integral of **H** along a closed curve not including a wire vanishes so that curl $\mathbf{H}=0$.

(c) The line integral of **H** along a contour including one of the wires is finite and has the value

$$\frac{4\pi}{10} I$$

where I is the total current through the wire (measured in the direction pointing out of the plane of the paper if the line integral is taken in the counter clockwise direction).

(d) At an infinite distance to either side of the wires the field **H** becomes uniform.

Consider the value of the component of this uniform field above and below the row of wires.

Take the line integral of **H** along the contour $ABCD$, the lines AB, CD being very long, perpendicular to line of the centers of the wires and drawn midway between the centers of the wires, 0_1 0_2, 0_3. The component of **H** normal to 0_1, 0_2 contributes nothing to the line integral because it is perpendicular to AD, BC and along the lines AB, CD, the contributions are equal and opposite. The component parallel to 0_1 0_3 is perpendicular to the lines AB, CD and hence its integral vanishes

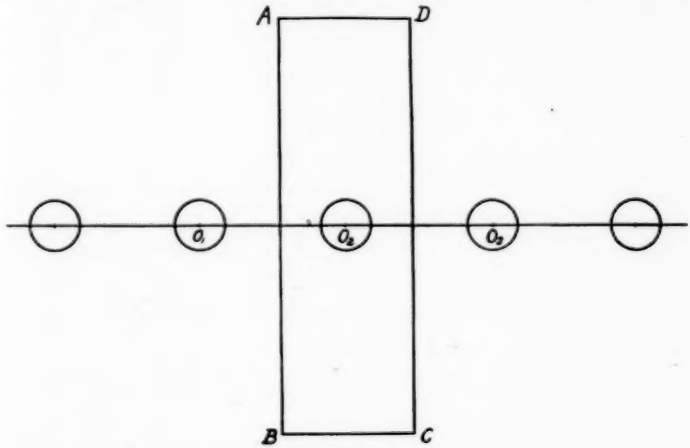

FIG. 3. *Diagram for discussion of magnetic field at infinity.*

along these lines. Denote the component parallel to 0_1 0_3 by H_p. If at AD its value is H_{p_1} (measured in the direction 0_1 0_2) and if at BC its value is H_{p_2} then

$$(-H_{p_1} + H_{p_2})AD = \frac{4\pi}{10} I$$

or

$$-H_{p_1} + H_{p_2} = \frac{4\pi I}{10h}$$

for $AD = h$ (the pitch of the winding).

Hence

$$H_{p_2} - \frac{2\pi I}{10h} = H_{p_1} + \frac{2\pi I}{10h} = H_p$$

This suggests that the general solution for H may be found by superposing the following three typical solutions.

(I) A solution satisfying the conditions (a), (b), (c), (d) and for which in addition the uniform field at infinity has no component normal to AD and has its tangential component such that the quantity H_p vanishes i.e. that

$$H_p = -\frac{2\pi I}{10h}$$

and that

$$H_p = +\frac{2\pi I}{10h}$$

The lines of force for this case are sketched in Fig. 4.

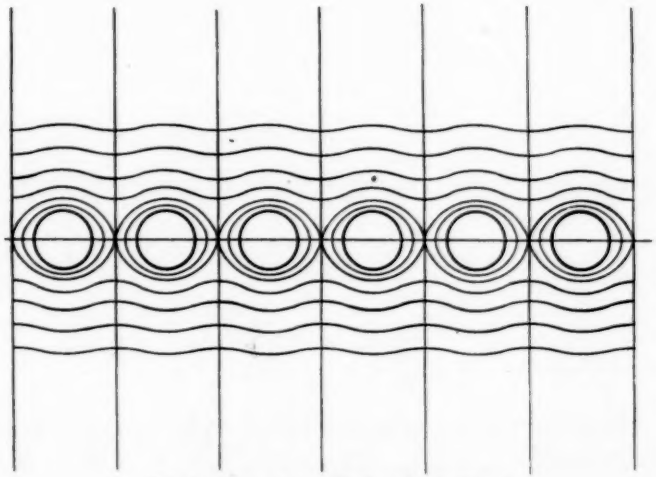

FIG. 4. *Lines of magnetic force for problem I.*

(II) A solution satisfying (a), (b) without the restriction of the contour of integration not including any one of the wires in its area and (d) with the additional restriction of having the component perpendicular to $0_1\ 0_3$ vanish at infinity. The lines of force for this case are sketched in Fig. 5.

(III) A solution satisfying (a), (b) without the restriction of the contour of integration not including any one of the wires in its area and (c) with the additional restriction of having the component parallel to the wires vanish at infinity.

The lines of force for this case are sketched in Fig. 6.

DISCUSSION OF PROBLEM I

It is shown in the Appendix that the expression

$$W(z) = \log z - \sum_{s=1}^{\infty} \frac{\mathfrak{C}_{2s}}{s} \frac{z^{2s} - z^{-2s}}{h^{2s}}$$

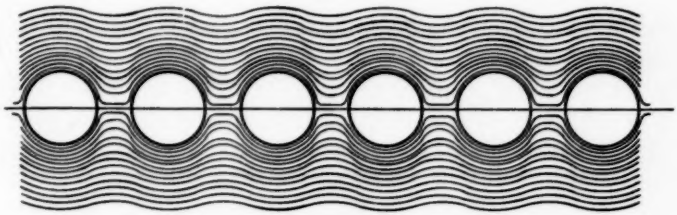

FIG. 5. *Lines of magnetic force for problem II.*

where $\mathfrak{C}_{2s}$ is the series which results on collecting terms of equal powers in h in the expression

$$A_{2s} - \sum_{s_1=1,2,..}^{\infty} 2A_{2s_1} A_{2s+2s_1} h^{-4s_1} \binom{2s+2s_1-1}{2s_1}$$

$$+ \sum_{s_1,s_2=1,2,..}^{\infty} A_{2s+2s_2} \cdot 2A_{2s_1+2s_2} \cdot 2A_{2s_1} h^{-4(s_1+s_2)} \binom{2s+2s_1-1}{2s_1} \binom{2s_1+2s_2-1}{2s_2}$$

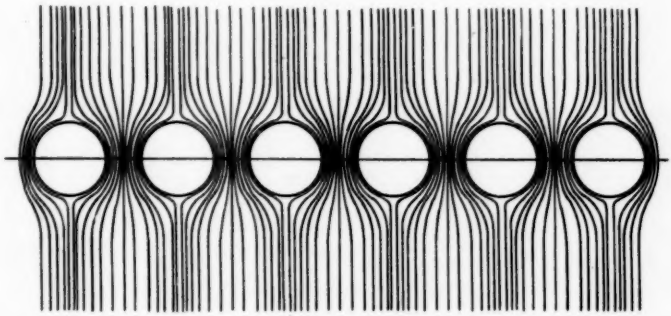

FIG. 6. *Lines of magnetic force for problem III.*

etc. where

$$A_{2s} = 1 + \frac{1}{2^{2s}} + \frac{1}{3^{2s}} + \ldots ; \quad \binom{m}{n} = \frac{\lfloor m}{\lfloor n \, \lfloor m-n}$$

so that

$$\mathfrak{C}_{2s} = A_{2s} - A_{2s+2} \cdot 2A_2 h^{-4} \binom{2s+1}{2}$$

$$+ h^{-8}\left\{ - A_{2s+4}\cdot 2A_4\binom{2s+3}{4} + A_{2s+2}\cdot 2A_4\cdot 2A_2\binom{2s+1}{2}\binom{3}{2}\right\}$$

$$+ h^{-12}\left\{ - A_{2s+6}\cdot 2A_6\binom{2s+5}{6} + A_{2s+4}\cdot 2A_6\cdot 2A_2\binom{2s+3}{4}\binom{5}{2}\right.$$

$$\left. + A_{2s+2}\cdot 2A_6\cdot 2A_4\binom{2s+1}{2}\binom{5}{4} - A_{2s+2}(2A_4)^2\,2A_2\binom{2s+1}{2}\binom{3}{2}^2\right\}$$

$$+ \text{etc.}$$

is likely to converge in the neighborhood of the unit circle and has the following properties.

Its real part vanishes over the unit circle. In the region in which $W(z)$ converges its real part represents the electrostatic potential which would be caused if each of the wires having the circles for cross sections carried an electrostatic charge of amount $-\frac{1}{2}$.

Again the imaginary part of $W(z)$ satisfies Laplace's equation and has no poles except inside the wires. The line integral of its gradient around one wire is 2π.

If the analytic continuation of $W(z)$ be taken its imaginary part degenerates into $-\pi h^{-1}z$ for $y = +\infty$ and into $\pi h^{-1}z$ for $y = -\infty$.

Thus the imaginary part of $2IW(z)/10$ is the magnetic potential for the problem (I). At the unit circle the imaginary part of $W(z)$ is

$$\theta - \sum_{s=1}^{\infty} 2\mathfrak{C}_{2s}\, s^{-1}\, h^{-2s} \sin 2s\theta$$

Its gradient is

$$1 - \sum_{s=1}^{\infty} 4\,\mathfrak{C}_{2s}\, h^{-2s} \cos 2s\theta$$

Consequently the magnetic intensity due to the field of problem (I) is

$$H_\theta^{(1)} = (2I/10)\left[1 - \sum_{s=1}^{\infty} 4\mathfrak{C}_{2s}\, h^{-2s} \cos 2s\theta\right]$$

Discussion of Problem II

It is shown in the appendix that if

$$\frac{a_{2s+1}}{2s+1} = b_{2s+1}$$

and if $\dfrac{b_{2s+1}}{b_1}$ is the result of ordering in equal powers of h the symbolical expansion

$$\frac{b_{2s+1}}{b_1} = -2h^{-(2s+2)}\left[A_{2s+2} - 2h^{-2}\sum_{s_1=1}^{\infty} A_{2s_1+2}\,A_{2s+2s_1+2}\binom{2s+2s_1+1}{2s_1} h^{-4s_1}\right.$$

$$\left. +(2h^{-2})^2\sum_{s_1,s_2=1,2,\ldots}^{\infty} A_{2s_1+2}\,A_{2s_1+2s_2+2}\,A_{2s+2s_1+2}\binom{2s+2s_1+1}{2s_1}\binom{2s_1+2s_2+1}{2s_2} h^{-4(s_1+s_2)} - \ldots\right]$$

i.e. if

$$-\frac{h^{2s+2}b_{2s+1}}{b_1} = A_{2s+2} - 2A_4A_{2s+4}\binom{2s+3}{2}h^{-6} - 2A_6A_{2s+6}\binom{2s+5}{4}h^{-10}$$

$$+2^2A_4A_6A_{2s+4}\binom{2s+3}{2}\binom{5}{2}h^{-12} - 2A_8A_{2s+8}\binom{2s+7}{2}h^{-14}$$

$$+2^2\left[A_4A_8A_{2s+6}\binom{2s+5}{4}\binom{7}{2} + A_6A_8A_{2s+4}\binom{2s+3}{2}\binom{7}{4}\right]h^{-16}$$

$$-\left[2A_{10}A_{2s+10}\binom{2s+9}{}+2^3A_4A^2{}_6A_{2s+4}\binom{2s+3}{2}\binom{5}{}^2\right]h^{-18}$$

$$+2^2\left[A_4A_{10}A_{2s+8}\binom{2s+7}{6}\binom{9}{2} + A_8A_{10}A_{2s+4}\binom{2s+3}{2}\binom{9}{6} + A_6A_{10}A_{2s+6}\binom{2s+5}{4}\binom{9}{4}\right]h^{-20}$$

$$- \ . \ . \ .$$

then provided

$$a_1 = b_1$$

is given the proper value, the magnetic intensity at the point having azimuth θ on the unit circle is

$$a_1 \sin\theta + a_3 \sin 3\theta + a_5 \sin 5\theta + \ . \ . \ .$$

in the counter clockwise direction while at infinity the field becomes a unit field, is parallel to the line of centers of the circles, and is directed from left to right.

The proper value of b_1 is given by the equation

$$1 = -2^{-1}b_1 - \sum_{s_1=1}^{\infty}(2s_1-1)\,b_{2s_1-1}\,A_{2s_1}\,h^{-2s_1}$$

i.e.

$$b_1 = -\left[\tfrac{1}{2} + \sum_{s_1=1}^{\infty}(2s_1-1)\left(\frac{b_{2s_1-1}}{b_1}\right)A_{2s_1}\,h^{-2s_1}\right]^{-1}$$

In this notation if the superposed field at infinity is to be H_{oz}

$$H_\theta^{(2)} = H_{oz}\,(a_1 \sin\theta + a_3 \sin 3\theta + a_5 \sin 5\theta + \ . \ . \ .)$$

is the magnetic field which accounts for the magnetic field superposed on the section along the axis.

DISCUSSION OF PROBLEM III

It is shown in the Appendix that if

$$\frac{a_{2s+1}}{2s+1} = \beta_{2s+1}$$

and if $\frac{\beta_{2s+1}}{\beta_1}$ is the result of ordering in powers of h^{-1} the symbolical expansion

$$\frac{h^{2s+2}\,\beta_{2s+1}}{2\beta_1} = A_{2s+2} + 2h^{-2}\sum_{s_1=1}^{\infty} A_{2s_1+2}\,A_{2s+2s_1+2}\,\binom{2s+2s_1+1}{2s_1}\,h^{-4s_1}$$

$$+(2h^{-2})^2\sum_{s_1,s_2=1,2,..}^{\infty} A_{2s_1+2}\,A_{2s_1+2s_2+2}\,A_{2s+2s_1+2}\,\binom{2s+2s_1+1}{2s_1}\binom{2s_1+2s_2+1}{2s_2}\,h^{-4(s_1+s_2)}$$

$$+ \ . \ . \ .$$

where β_1 is given the value

$$\beta_1 = \left[-\tfrac{1}{2} + \sum_1^{\infty}(2s-1)\,(\beta_{2s-1}/\beta_1)\,A_{2s}\,h^{-2s}\right]^{-1}$$

then the magnetic intensity at the wire becomes

$$-a_1\cos\theta - a_3\cos 3\theta - a_5\cos 5\theta - \ . \ . \ .$$

the magnetic intensity at infinity becoming uniform, parallel to the axis of y and equal numerically to (1).

Thus if the superposed field in the direction perpendicular to OY is H_{oy}, then

$$H_\theta^{(3)} = -H_{0y}\,(a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + \ . \ . \ .)$$

is the field at the wire.

The resultant field is then

$$H_\theta = H_\theta^{(1)} + H_\theta^{(2)} + H_\theta^{(3)} = (2I/10)\,\left[1 - \sum_{s=1}^{\infty} 4\mathfrak{C}_{2s}\,(\cos 2s\,\theta)h^{-2s}\right]$$

$$+H_{ox}\sum_{s=1}^{\infty} a_{2s-1}\sin(2s-1)\theta - H_{oy}\sum_{s=1}^{\infty} a_{2s-1}\cos(2s-1)\,\theta$$

The current flowing per unit length is

$$u = \frac{10}{4\pi}H\theta$$

It is clear from the above expression for H_θ that

$$\int_0^{2\pi} u\,d\theta = I$$

Application to the Calculation of Resistance

If now it is desired to apply these results to the calculation of resistance the following reasoning may be used.

As explained the current flows in a thin skin, the thickness of the skin being independent of the density of current. If the resistance of a portion of this skin bounded by a unit length of the circumference be σ the power lost in an element ds is

$$\sigma u^2 ds$$

and the total power lost is

$$\int_0^{2\pi} \sigma u(\theta)^2 d\theta$$

The total current flowing is I.
Hence the resistance is

$$R = \int_0^{2\pi} \sigma u(\theta)^2 I^{-2} d\theta$$

But if the wire were straight the current density u would be

$$u = I/(2\pi)$$

Hence for this case the resistance would be

$$R_0 = \frac{\sigma}{2\pi}$$

Consequently the ratio of the resistance of the coil to the resistance of the wire of the coil when straight is [on substituting

$$u = \frac{10}{4\pi} H\theta]$$

$$\frac{R}{R_0} = 1 + \tfrac{1}{2} \sum_{s=1}^{\infty} (4\mathfrak{C}_{2s}\, h^{-2s})^2 + \left(\frac{10 H_{oz}}{2I}\right)^2 \sum_{s=1}^{\infty} \frac{a_{2s-1}^2}{2}$$

$$+ \left(\frac{10 H_{oy}}{2I}\right)^2 \sum_{s=1}^{\infty} \frac{a_{2s-1}^2}{2}$$

It is remarkable that at high frequencies as well as at low only the quantities H_{oz}^2, H_{oy}^2 must be integrated in order to make the correction for the finite length of the solenoids.

SPECIAL CASE

It is especially interesting to investigate the case when $h = 2$, because this case has been treated by Sommerfeld and also because solutions of Problems (I) and (II) separately are capable of giving an answer to this case.

The reason for the identity between the solutions of (I) and (II) for this case lies in the fact that the wires touch. Thus the wires

separate all space into two parts (form a wall) and their cross sections separate the plane considered into two parts—one lying on one side of the wires and the other on the other side. Both (I) and (II) make the field uniform at infinity and tangential to the wires. Hence the only difference between the two solutions may be given by a constant factor.

If now $H_{oz}=H_{oy}=0$ the above expression becomes

$$\frac{R}{R_0} = 1 + 8\sum_{s=1}^{\infty} (\mathfrak{C}_{2s}\, h^{-2s})^2$$

This formula corresponds then to the case when the current distribution is symmetrical with respect to the axis of X. The above resistance ratio when computed for this case must be then only one-half of the resistance ratio in the case when no current flows on the outside of the coil (i.e. above OX) and all is crowded on the inside, for the inside and outside may be considered as two circuits joined in parallel.

Using the formulas developed and the expression for A_{2s}, in Bernoulli numbers, viz:

$$A_{2s} = \frac{(-)^{s-1}(2\pi)^{2s}\, B_{2s}}{2(2s)!}$$

giving

$$A_2 = \frac{\pi^2}{6},\ A_4 = \frac{\pi^4}{90},\ A_6 = \frac{\pi^6}{945},\ A_8 = \frac{\pi^8}{9450},\ \text{etc.}$$

it is found that

$$\mathfrak{C}_2 = 1.190$$
$$\mathfrak{C}_4 = -0.23$$

so that for the case of $H_{oz}=H_{oy}=0$

$$\frac{R}{R_0} = 1 + 8\left(\frac{(1.190)^2}{16} + \frac{(0.06)^2}{256} + \ldots\right) = 1.71$$

Hence for Sommerfeld's case the resistance ratio is

$$3.42$$

A different way of obtaining the same result is afforded by the solution of Problem II. It is simplest in this case to develop the resistance formula independently.

The quantity

$$a_1 \sin\theta + a_3 \sin 3\theta + a_5 \sin 5\theta + \ldots$$

was seen to be proportional to the magnetic field and therefore to the linear current density u. On a proper scale the current density u is then

$$u(\theta) = \sin\theta + \frac{a_3}{a_1}\sin 3\theta + \frac{a_5}{a_1}\sin 5\theta + \ldots$$

This current flows only on the inside of the coil. Its total value is then numerically

$$\int_0^{\pi} u(\theta)\, d\theta = 2\left(1 + \frac{a_3}{3a_1} + \frac{a_5}{5a_1} + \ldots\right)$$

The amount of power lost is

$$\sigma \int_0^{\pi} u(\theta)^2\, d\theta = \frac{\pi\sigma}{2}\left(1 + \left(\frac{a_3}{a_1}\right)^2 + \left(\frac{a_5}{a_1}\right)^2 + \ldots\right)$$

and the resistance

$$R = \frac{\pi\sigma}{8} \frac{1 + \left(\frac{a_3}{a_1}\right)^2 + \left(\frac{a_5}{a_1}\right)^2 + \ldots}{\left(1 + \frac{a_3}{3a_1} + \frac{a_5}{5a_1} + \ldots\right)^2}$$

The resistance of the same wire if it were straight would be

$$R_0 = \frac{\sigma}{2\pi}$$

Hence

$$\frac{R}{R_0} = \frac{\pi^2}{4} \frac{1 + \left(\frac{a_3}{a_1}\right)^2 + \left(\frac{a_5}{a_1}\right)^2 + \ldots}{\left(1 + \frac{a_3}{3a_1} + \frac{a_5}{5a_1} + \ldots\right)^2}$$

A computation gives

$$-\frac{b_3}{b_1} = 0.101 \qquad \frac{a_3}{a_1} = -0.304$$

$$-\frac{b_1}{b_1} = 0.013 \qquad \frac{a_5}{a_1} = -0.065$$

whence

$$\frac{R}{R_0} = \frac{\pi^2}{4} \frac{1.097}{(0.889)^2} = 3.42$$

$$\left(\text{The number } 0.889 = \frac{\pi^2}{16 \log 2}\right)$$

The value obtained by Sommerfeld is 3.73.

His method is essentially as follows. The theory of elliptic modular functions gives him an expression for the magnetic field H_θ. The linear current density is proportional to H_θ. The resistance is

$$R = \frac{\int_0^\pi \sigma H_\theta^2 \, d\theta}{\left(\int_0^\pi H_\theta \, d\theta\right)^2}$$

The resistance of the same wire if it were straight would be

$$R = \frac{\sigma}{2\pi}$$

Hence

$$\frac{R}{R_0} = \frac{2\pi \int_0^\pi H_\theta^2 \, d\theta}{\left(\int_0^\pi H_\theta^2 \, d\theta\right)^2}$$

He then evaluates both of these integrals graphically. The value of the integral in the denominator is known to him from theoretical considerations and is checked by the graphical method. For the integral in the numerator he finds

$$0.378 \times 2\pi$$

The author had redrawn the curve (See Fig. 7) for H_θ^2 and has found by measuring the area of the curve by Simpson's rule and also by an intergraph that it is

$$0.345 \times 2\pi$$

It thus appears that at this point Sommerfeld's calculation is in error. Correcting for this his result must be

$$\frac{0.345}{0.378} \times 3.73 = 3.41$$

Considering the fact that an inaccuracy of 2% in the area is possible in drawing the curve this may be considered as in good agreement with the calculations given here.

The expansions for the coefficients have been carried further than are here indicated and using these expansions values of $\frac{R}{R_0}$ were calculated for value of h nearly equal to 2. This was desirable because the expansions are semi-convergent at $h=2$. They were also summed at

$h=2$ by the Cèsaro method. The relation between $\frac{R}{R_0}$ and h is plotted in Fig. 7a.

The calculation of H_{ox} and H_{oy} can be carried out by replacing the shaded cylinders GE, CA by their equivalent polarized distributions. For a current of 10 amperes per unit length we deal with a unit surface density of magnetic charge over EF and AB, and a negative unit

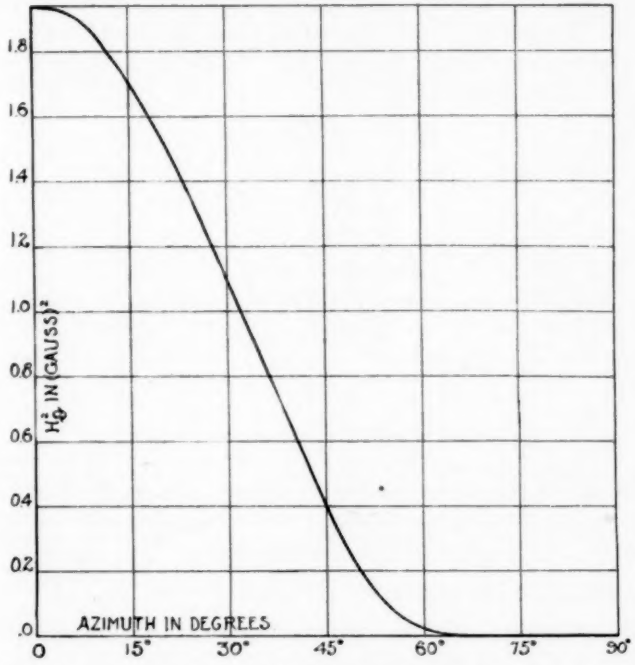

FIG. 7. *Variation of square of magnetic intensity along circumference of the wire's cross section according to Sommerfeld's calculation.*

density of charge over GH and CD. It is seen by integration that a circular disc of radius a having a uniform density of charge and situated at EF gives at P a component of H in the direction of the axis $= \pi - \frac{2x}{\sqrt{x^2+4a^2}} F\left(\frac{2a}{\sqrt{x^2+4a^2}}\right)$ where F is the complete elliptic integral of the first kind and where a is the radius of EF while $EP=x$.

The value of H_{oy} the writer first derived by a long integration but then in a conversation with Professor Ehrenfest a short derivation was

evolved as follows. The force on a turn of wire in the direction of OX if this wire also carries 10 amp. is

$$F_x = -2\pi a H_{oy} = -\frac{\partial}{\partial x} M(x) = -M(l_1) + M(l_2)$$

where $M(x)$ = mutual inductance between wire and solenoid and where $M(l)$ is the expression for the mutual inductance between two coaxial circles of radius a a distance l apart (See Jeans El. & Mag. 4th ed. p. 443)

Dr. F. W. Grover of the Bureau of Standards had derived this result independently. He has also very kindly checked many of the calculations here given.

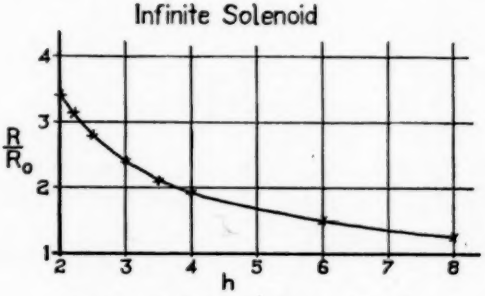

FIG. 7a. *Variation of resistance with pitch.*

Professor Sommerfeld on hearing of the results here given has gone over his calculation and has obtained a value agreeing with the values of Abraham, Bloch and Bloch, and of the writer.

Appendix I. *Solution of Problem I.*

Statement of problem. Let a number of unit circles be given with centres at $(n\ h,\ o)$ $(n = 0 \pm 1,\ \pm 2, \text{——})$.

Find such a field **H** that everywhere outside the wires H is finite and continuous, that *div* **H** = 0, and that the line integral of **H** around any closed curve including any one of the circles should vanish. The line integral of **H** taken counter clockwise around a closed curve including one of the circles must be 2π. For $= y + \infty$ the field is to become uniform, it must be parallel to OX and it must be pointed in the negative direction of the axis of x. For $y = -\infty$ the field is also to become uniform, it must become parallel to the axis of x and it must point in the positive direction of the axis of x.

The field **H** may be derived from a multivalued potential.

Let $z=x+jy$, $j=\sqrt{-1}$ since div$\mathbf{H}=0$, if the potential from which $\mathbf{H}$ is derived be U

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0$$

Thus U may be considered as the imaginery part of some function of z, say

$$W(z)$$

Thus

$$W(z) = V(x\ ,\ y) + jU(x\ ,\ y)$$

The functions V, U satisfy the Cauchy conditions

$$\frac{\partial V}{\partial x}=\frac{\partial U}{\partial y}$$

$$\frac{\partial V}{\partial y}=-\frac{\partial U}{\partial x}$$

Hence the line integral of grad U around the unit circle is equal to the flux outward of the grad V through that unit circle. But the problem states that the first of these quantities is 2π. Therefore, the second is also 2π

i.e.

$$\int_0^{2\pi}\left(\frac{\partial V}{\partial \rho}\right)_{\rho=1} d\theta = 2\pi$$

Similarly for $y=+\infty$, $\frac{\partial U}{\partial y}=0$, and $\frac{\partial U}{\partial x}$ is constant and negative. Therefore $\frac{\partial V}{\partial y}$ is constant and positive while $\frac{\partial V}{\partial x}=0$. Similarly for $y=-\infty$, $\frac{\partial V}{\partial y}$ is constant and negative while $\frac{\partial V}{\partial x}=0$. Thus grad V at infinity points away from the circles. The total flux of grad V through a curve enclosing one circle is 2π. Further

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

Thus V is the electrostatic potential corresponding to the case of each wire carrying a total charge $-\frac{1}{2}$, the dielectric constant of the medium being 1, the field caused being symmetrical with respect to the plane through the axes of the wires, and each wire being itself an equipotential.

Since U is multiple valued and V is not it will be convenient to discuss V, and then by means of the function $W(z)$ to find U.

V has been defined throughout the exterior of the circles $|z-nh|=1$ $(n=0,\pm1,\pm2,$ ——). Its definitions will now be extended also to the interior of these circles.

Invert all of the plane z in the circle $|z|=1$. The process of inversion sends the outside of the circle of inversion into the inside and the inside into the outside. Thus all the circles $|z-nh|=1$ $(n=\pm1,\pm2$ ——) which are on the outside are sent into corresponding circles inside. The centres of these circles are clearly on the axes of reals. A sketch of these circles is given on Fig. (8).

The function V will next be defined throughout the shaded region of this figure i.e., throughout the region inside the circle $|z|=1$ and outside the circles which are the inverses of $|z-nh|=1$. Subtract from V a constant so as to make it zero at $|z|=1$.

If now P^{-1} is a point in the shaded region and if P is its inverse the value of V at P^{-1} is defined to be the negative of the value of V at P.

Since V is zero over $|z|=1$. V as now defined is continuous at $|z|=1$. Further $\left(\frac{\partial V}{\partial\theta}\right)_{\rho=1}$ vanishes. Therefore, the tangential component of *grad*. V is continuous. Also the normal component of *grad* V is continuous.

In fact if (ρ,θ) is a point outside the circle, its inverse is $(\frac{1}{\rho},-\theta)$. V outside is a function of ρ and θ say

$$V_e=\Psi(\rho,\theta)$$

By definition V inside is

$$V_i=-\Psi\left(\frac{1}{\rho},-\theta\right)=\chi(\rho,\theta)$$

In order to show that the normal component of grad V is continuous it is sufficient to show that

$$\left(\frac{\partial V_e}{\partial\rho}\right)_{\rho=1}=\left(\frac{\partial V_i}{\partial\rho}\right)_{\rho=1}$$

i.e., that

$$\left(\frac{\partial\Psi(\rho,\theta)}{\partial\rho}\right)_{\rho=1}=\left(\frac{\partial\chi(\rho,\theta)}{\partial\rho}\right)_{\rho=1}$$

But

$$\left(\frac{\partial\chi(\rho,\theta)}{\partial\rho}\right)_{\rho=1}=\left(\frac{\partial\Psi(\rho,-\theta)}{\partial\rho}\right)_{\rho=1}$$

and by symmetry $\Psi(\rho, \theta) = \Psi(\rho, -\theta)$

$$\left(\frac{\partial V_e}{\partial \rho}\right)_{\rho=1} = \left(\frac{\partial V_i}{\partial \rho}\right)_{\rho=1}$$

Thus V, $\frac{\partial V}{\partial \theta}$, $\frac{\partial V}{\partial \rho}$ as defined are continuous at the unit circle.

Just as V was defined through the shaded region in the unit circle it could have been defined through similar regions in the circles $|z-nh| = 1$. These regions when inverted in $|z| = 1$ go into regions inside the circles of Fig. 8.

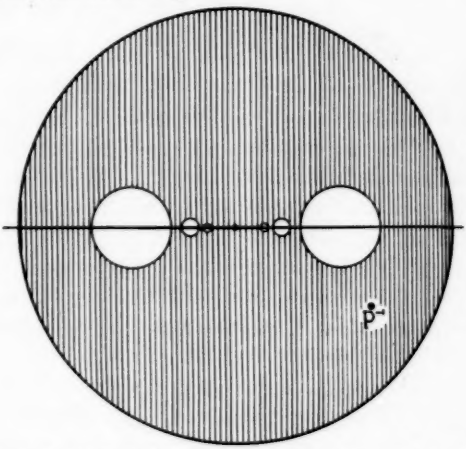

FIG. 8. *Illustration of Method of Inversion used in solution of potential problem I.*

Call the original set of circles $C_n^{(1)}$. Let the result of inverting of all of the $C_n^{(1)}$ in one be $C_n^{(2)}$. Let the result of inverting all of the $C_n^{(2)}$ in one of the $C_n^{(1)}$ be $C_n^{(3)}$ etc. Inside every $C_n^{(1)}$ an infinite number of $C_n^{(2)}$ is situated. V has been defined throughout the region bounded by the $C_n^{(2)}$ and by the $C_n^{(1)}$. On inverting in a $C_n^{(1)}$ this region goes into a region inside a $C_n^{(2)}$ and bounded on its inside by an infinity of $C_n^{(3)}$. It is now obvious how V should be defined inside such a region in order to be continuous and in order to have its first derivatives continuous. This condition is clearly satisfied by making V equal to the negative of its value at the inverse of the point in question.

It is thus clear that the extension of the definition of V to the region bounded by $C_n^{(2)}$ and $C_n^{(1)}$ leads to an extension to the regions bounded by $C_n^{(3)}$ and $C_n^{(2)}$, that this in turn leads to an extension of the definition of V to regions bounded by a $C_n^{(3)}$ and an infinity of $C_n^{(4)}$ etc.

It is clear that this process can be continued indefinitely. It is also readily proved that the maximum radius of a $C_n^{(m)}$ decreases indefinitely as m increases indefinitely.

Hence if (x, y) be the Cartesian coordinates of a point and if $y \neq 0$, it is always possible no matter how small y may be to find such an m that the maximum radius of a $C_n^{(m)}$ is less than y and that (x, y) therefore lies inside a region in which V has been defined.,

Further it is clear that the point outside the $C_n^{(1)}$ to which (x, y) can have its origin traced has also $y \neq 0$, and $y \neq \infty$. Therefore, V and its derivatives are finite at that point and consequently the same is true for the point (x, y).

Thus V may be defined so that it itself and its first derivatives are finite and continuous at all points not on the axis of reals.

Again if $h > 2$ the largest of the $C_n^{(2)}$ inside a $C_n^{(1)}$ does not touch the $C_n^{(1)}$. There is therefore a segment left on the axis of reals and in that segment V and its derivatives are finite and continuous. If $h = 2$ this is not true. But this case does not require any treatment except as the limiting case of $h > 2$ because of $h = 2$ the wires touch.

Therefore, for all practical cases each $C_n^{(1)}$ is situated in a region in which V and its first derivatives are finite and continuous.

The function $W(z)$ may also be defined in this region by means of the function U. The definition will be given so as to have U a single valued function. This is accomplished if a cut is made in the region along the negative of the axis of reals and then U is defined by the relations

$$\frac{\partial V}{\partial x} = \frac{\partial U}{\partial y}$$

$$\frac{\partial V}{\partial y} = \frac{-\partial U}{\partial x}$$

for these relations give the line integral of U along any curve in the region and the region as defined is a simply connected region.

This defines U and hence $W(z)$ save as to a constant. $W(z)$ thus defined is analytic in the region. However, it is not analytic throughout the region if that region is uncut.

The boundaries of the region can be made to be two concentric circles. To the annulus included by the circles Laurent's Theorem could be applied for a function which is analytic throughout the annulus. Now $W(z)$ is not analytic throughout the annulus because on going around the annulus once the value of $W(z)$ is increased by

$$2\pi j$$

The function *log z* is also not analytic throughout the annulus and its value also increases by $2\pi j$, on going once around the annulus. However, it is analytic in the annulus.

It is clear then that

$$W(z) - \log z$$

is analytic throughout the annulus. It must, therefore, be possible to find a Laurent expansion for it in the annulus.

The constant term of the expansion is of no interest and may be thrown out. The remaining part must be such as to be constant over the unit circle. On using the Laurent expansion explicitly and transforming the result in a Fourier series by the formula

$$\epsilon^{j\theta} = \cos\theta + j \sin\theta$$

one finds that if this constant value of $W(z) - \log z$ should be zero, then the constant term in Laurent's expansion is zero and also on using the fact that the solution is symmetrical with respect to the axes of reals and pure imaginaries one finds that

$$W(z) = \log z - \Sigma\, \mathfrak{C}_{2s}\, s^{-1}\, h^{-2s}\, (z^{2s} - z^{-2s})$$

where all $\mathfrak{C}_{2s}$ are real.

It now remains to determine the $\mathfrak{C}_{2s}$. This can be done by confining one's attention to V. The expression for V is

$$V = \log \rho - \sum_{s=1}^{\infty} \mathfrak{C}_{2s}\, s^{-1}\, h^{-2s}\, (\rho^{2s} - \rho^{-2s}) \cos 2s\theta \qquad \text{(I)}$$

It is convenient to think of V as the potential due to charges distributed over the surface of the wires. The charge density σ is given by

$$\sigma = -\frac{1}{4\pi}\left(\frac{\partial V}{\partial \rho}\right)_{\rho=1} = -\frac{1}{4\pi}\left[1 - \sum_{s=1}^{\infty} 4\, \mathfrak{C}_{2s}\, h^{-2s} \cos 2s\theta\right] \qquad \text{(II)}$$

As known from the theory of the logarithmic potential the potential V is save as to a constant obtained by integrating

$$\int (-)\, 2\sigma \log r ds$$

where ds is an element of arc at which the surface density is σ and whose distance from the point at which the potential is computed is r.

Now over the unit circle the potential V is to be zero. Hence in order to get the right value of V in the calculation that follows it will be sufficient to add such a constant to the potential contributed by any one of the wires that the potential should vanish at $|z| = 1$.

The wire whose section is $|z+nh| = 1$, contributes to the potential at the point $\rho=1$, $\theta=\theta_0$ the amount

$$\frac{1}{2\pi}\int^{2\pi}\left[1-\sum_{s=1}^{\infty}4\mathfrak{C}_{2s}h^{-2s}\cos 2s\theta\right]\log\sqrt{(nh-\cos\theta+\cos\theta_0)^2+(\sin\theta-\sin\theta_0)^2}\,d\theta$$

the reservation being made that a proper constant may be added to this.

Letting

$$r_n = \sqrt{(nh+\cos\theta_0)^2+\sin^2\theta_0}$$

$$\tan\theta_n = \frac{\sin\theta_0}{nh+\cos\theta_0}$$

and remembering that

$$\log\sqrt{r_n^2-2r_n\cos\theta+1} = \log r_n-\sum_{s=1}^{\infty}s^{-1}r_n^{-s}\cos s\theta$$

The above integral reduces to

$$\log r_n+\sum_{s=1}^{\infty}s^{-1}r_n^{-2s}h^{-2s}\mathfrak{C}_{2s}\cos 2s\theta$$

To this now

$$\log\left|\frac{1}{nh}\right| - \sum_{s=1}^{\infty}\mathfrak{C}_{2s}s^{-1}h^{-2s}(nh)^{-2s}$$

is added so as to keep the average value at zero. Thus all sections with the exception of the first give

$$\sum_{n=-\infty}^{n=+\infty}\log\frac{r_n}{|nh|}+\sum_{n=-\infty}^{n=+\infty}{}'\sum_{s=1}^{\infty}s^{-1}h^{-2s}\mathfrak{C}_{2s}\left(r_n^{-2s}\cos 2s\theta_n-(nh)^{-2s}\right)$$

the accent on the summation showing that the value $n=0$ is omitted

To this it is necessary to add the contribution to the potential due to the charges on $|z| = 1$. This may be obtained by direct integration or simpler as follows.

Consider the potential given by $\rho^m\cos m\theta$ inside the unit circle and by $-\rho^{-m}\cos m\theta$ outside. Both satisfy Laplace's equation, both are finite and continuous everywhere and have finite and continuous derivatives. Further the value of both at the unit circle is $\cos m\theta$. Thus they represent the potential which would be caused by a proper distribution of charge density on the unit circle. This charge density is

$$\frac{m\cos m\theta}{2\pi}$$

Hence it is clear that the charge density

$$\sigma = -\frac{1}{4\pi}\left[1-\sum_{s=1}^{\infty} 4h^{-2s}\mathfrak{C}_{2s}\cos 2s\theta\right]$$

may be considered as giving rise to a potential which at the unit circle is

$$\sum_{s=1}^{\infty} s^{-1}\, h^{-2s}\,\mathfrak{C}_{2s}\cos 2s\theta$$

The average value of this over the unit circle is zero. Hence the undetermined constant which in general could be added to this expression, is zero.

Consequently at the point having azimuth θ_0 on the unit circle the potential (which according to the conditions of the problem must vanish) is

$$V = \sum_{s=1}^{\infty} s^{-1}\, h^{-2s}\,\mathfrak{C}_{2s}\cos 2s\theta_0 + \sum_{n=-\infty}^{n=+\infty}{}' \log\frac{r_n}{|nh|}$$
$$+ \sum_{n=-\infty}^{n=+\infty}{}' \sum_{s=1}^{\infty} s^{-1}\, h^{-2s}\,\mathfrak{C}_{2s}\,(r_n^{-2s}\cos 2s\theta_n - (nh)^{-2s})$$

The condition to be expressed is that

$$V = 0$$

In expressing this condition it is convenient to reverse the order of summation in the last term on the right in the expression for V.

If now one should write

$$z = e^{j\theta_0}$$

it follows easily that

$$\sum_{n=-\infty}^{n=+\infty}{}' \left(\frac{1}{z-nh}+\frac{1}{nh}\right) = -2\sum_{s=1}^{\infty} A_{2s}\, z^{2s-1}\, h^{-2s}$$

where

$$A_{2s} = 1 + \frac{1}{2^{2s}} + \frac{1}{3^{2s}} + \ldots$$

Differentiating this as to z it is found that

$$\sum_{n=-\infty}^{n=+\infty}{}' \left(\frac{1}{z-nh}\right)^{2s} = 2\left[A_{2s}h^{-2s} + \binom{2s+1}{2s-1} A_{2s+2}h^{-(2s+2)}z^2 \right.$$
$$\left. + \binom{2s+3}{2s-1} A_{2s+4}h^{-(2s+4)}z^4 + \ldots\right]$$

Whence on taking the real part

$$\sum_{n=-\infty}^{n=+\infty}{}' (r_n^{-2s}\cos 2s\theta_n-(nh)^{-2s})=2\sum_{p=1}^{\infty}\binom{2s+2p-1}{2s-1}A_{2s+2p}\,h^{-2s-2p}\cos 2p\theta_0$$

Therefore

$$\sum_{n=-\infty}^{n=+\infty}{}'\sum_{s=1}^{\infty}s^{-1}h^{-2s}\mathfrak{C}_{2s}(r_n^{-2s}\cos 2s\theta_n-(nh)^{-2s})$$
$$=\sum_{s=1}^{\infty}2s^{-1}h^{-2s}\mathfrak{C}_{2s}\sum_{p=1}^{\infty}\binom{2s+2p-1}{2s-1}\frac{A_{2s+2p}\cos 2p\theta_0}{h^{2s+2p}}$$

Consequently if it be considered as a double series in s and p this double series if absolutely convergent may be, first summed as to s and then as to p.

Substituting this series into the expression $V=0$ and replacing in this

$$\sum_{n=-\infty}^{n=+\infty}{}'\log\sqrt{\frac{(nh+\cos\theta_0)^2+\sin^2\theta_0}{n^2h^2}}$$

by its value viz:

$$\sum_{s=1}^{\infty}s^{-1}h^{-2s}A_{2s}\cos 2s\theta_0$$

and then equating to zero the coefficient of $\cos 2s\theta_0$ it is found that

$$\mathfrak{C}_{2s}=A_{2s}-\sum_{s_1=1}^{\infty}2\mathfrak{C}_{2s_1}A_{2s+2s_1}\binom{2s+2s_1-1}{2s_1}h^{-4s_1} \qquad (1)$$

Consider now the distribution of potential and the shape of the lines of force for the case when the distance h becomes very large. The distribution of potential for this case is obviously precisely similar to that obtained by keeping the distance between the centers of the circles fixed and diminishing their radius indefinitely. As this is done the problem degenerates into that of finding the lines of force inside the area of the infinite rectangle *ABCEOFG*. (See Fig. 9), when at the center O of the rectangle an indentation *CDE* is made in the form of an infinitesimal semicircle and when the contour *ABC*, *EFG* is to be a line of force while the infinitesimal semicircle is to be an equipotential. If *CDE* shrinks to a point, the distribution of potential over the semicircle *BHF* approaches a definite limit.

On the other hand, the expansion for $W(z)$ holds for all points on *BHF* because *OF* is smaller than the distance from O to the reflection

of E in GF. But the circle BHF corresponds to $\rho = h$ so that the value of V on BHF is obtained (for very large values of h) as

$$\log\left(\frac{h}{2}\right) - \sum_{s=1}^{\infty} s^{-1}\, 2^{-2s}\, \mathfrak{C}_{2s} \cos 2s\theta$$

This shows that the distribution over BHF in the limit depends on $\frac{\mathfrak{C}_{2s_1}}{\mathfrak{C}_{2s_2}}$ and since this distribution approaches a definite limit, this shows that

$$\lim_{h=\infty} \frac{\mathfrak{C}_{2s_1}}{\mathfrak{C}_{2s_2}}$$

is finite (distinct from zero).

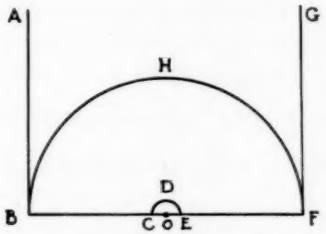

FIG. 9. *Degenerate case of problem I.*

Dividing (1) by $\mathfrak{C}_{2s}$ and passing to the limit it follows that

$$\lim_{h=\infty} \mathfrak{C}_{2s} = A_{2s}$$

Again by (1)

$$h^4(\mathfrak{C}_{2s} - A_{2s}) = -2\mathfrak{C}_2 A_{2s+2}\binom{2s+1}{2} - 2\mathfrak{C}_4 A_{2s+4}\binom{2s+3}{4} h^{-4} + \ldots$$

and

$$\lim_{h=\infty} h^4\,(\mathfrak{C}_{2s} - A_{2s}) = -2A_2 A_{2s+2}\binom{2s+1}{2}$$

This shows then that, considered as a function of h^{-4}, $\mathfrak{C}_{2s}$ has a derivative at 0 and that the value of this derivative is

$$-2A_2\, A_{2s+2}\binom{2s+1}{2}$$

It, therefore, is natural to assume that $\mathfrak{C}_{2s}(h)$ can be expanded in powers of h^{-4}.

If this should actually be the case the computation of $\mathfrak{C}_{2s}(h)$ should be easy in virtue of (1). In fact the whole expansion could then be obtained by collecting equal powers of h in the symbolical expansion.

$$\mathfrak{C}_{2s}=A_{2s}-\sum_{s_1=1}^{\infty}2A_{2s_1}A_{2s+2s_1}\binom{2s+2s_1-1}{2s_1}h^{-4s_1}$$

$$+\sum_{s_1,s_2=1}^{\infty}2A_{2s_1}\cdot 2A_{2s_1+2s_2}\cdot 2A_{2s+2s_1}\binom{2s_1+2s_1-1}{2s_2}\binom{2s_1+2s_2-1}{2s_2}h^{-4(s_1+s_2)}+\ \ldots$$

In such a manner one gets

$$\mathfrak{C}_{2s}=A_{2s}-A_{2s+2}\cdot 2A_2\binom{2s+1}{2}h^{-4}+[-A_{2s+4}\cdot 2A_4\binom{2s+3}{4}$$

$$+A_{2s+2}\cdot 2A_4\cdot 2A_2\binom{2s+1}{2}\binom{3}{2}]h^{-8}+[-A_{2s+6}\cdot 2A_6\binom{2s+5}{6}$$

$$+A_{2s+4}\cdot 2A_6\cdot 2A_2\binom{2s+3}{4}\binom{5}{2}+A_{2s+2}\cdot 2A_6\cdot 2A_4\binom{2s+1}{2}\binom{5}{4}$$

$$-A_{2s+2}(2A_4)^2\,2A_2\binom{2s+1}{2}\binom{3}{2}^2]h^{-12}+\ \ .\ .$$

This then gives the solution of the problem provided $\mathfrak{C}_{2s}$ may be expanded in powers of $\frac{1}{h}$. This solution gives of course for V and σ expressions (I) and (II).

It remains, therefore, to decide whether it is legitimate to expand $\mathfrak{C}_{2s}$ in a descending power series in h.

If this power series is convergent it must satisfy (1). It also satisfies it in such a way as to degenerate into A_{2s} when $h>\infty$. Within the circle of convergence the power series is uniformly convergent and hence is a continuous function of h. Its value $\mathfrak{C}'_{2s}$ coincides with the actual value of $\mathfrak{C}_{2s}$ when $h>\infty$. The same is true for its first derivative and as can easily be shown also for the other derivatives. If then Taylor's Theorem should be applied to the function $\mathfrak{C}_{2s}(h)$ then it will be clear that the remainder term vanishes and that hence

$$\mathfrak{C}_{2s}'=\mathfrak{C}_{2s}$$

Thus it is sufficient to show that the expansion obtained for is convergent.

If the expansion is convergent for one value of h it is *a fortiori* convergent for a larger value of h. Hence if it could be shown that the expansion is convergent for $h=2$ it would thereby be shown that it is convergent for all values of $h>2$.

Again even if $\mathfrak{C}_{2s}$ is divergent for $h=2$ it is not necessarily divergent for values of $h>2$. In particular the question arises whether a suffi-

ciently large value can be found for h to make its expansion convergent. If such a value cannot be found this means that at $\frac{1}{h}=0$ the function $\mathfrak{C}_{2s}-A_{2s}$ cannot be expanded in a power series in $\frac{1}{h}$. In virtue of (1), however, expansions in fractional powers of $\frac{1}{h}$ are made impossible and similarly the occurrence of a logarithmic or any other term but one in $\left(\frac{1}{h}\right)$ is made impossible. Thus for sufficiently small values of $\left(\frac{1}{h}\right)$, $\mathfrak{C}_{2s}$ must be given by the power series written for it.

A general method for solving problems on skin effect[4] makes it probable that the increase in the resistance does not involve $\left(\frac{1}{h}\right)$ in the denominator of any of its formulas. Since the quantities $\mathfrak{C}_{2s}$ occur in the numerator of the resistance formulas and since the two results for the resistance must be identical it follows that the $\mathfrak{C}_{2s}$ generally is likely to be expressible as a power series in $\frac{1}{h}$. For if this were not the case the resistance could be defined for complex values of $\frac{1}{h}$ by its continuation and would have a pole corresponding to the denominator in one case and not the other.

Appendix II

Solution of Problem II

Statement of Problem

For the same system of circles as in Problem I, find such a field $\mathbf{H}$ that div $\mathbf{H}=0$, that the line integral of $\mathbf{H}$ around any closed curve (including or not including any of the circles) should vanish and that at an infinite distance to either side of the circles the field should be uniform and parallel to OX.

Solution.—A solution of this problem will be found if such a distribution of charge over each cylinder will be found that the potential caused by all of these charges at a point on the cylinder is $\sin\theta$. In fact this potential may be regarded as the imaginary part of some function $W(z)$ say

$$Im\ W(z)$$

[4] See G. Breit, "A Method of Solving Problems on Skin Effect." Forthcoming publication.

This function is such that

$$Im\ (z - W(z)) = 0$$

because at the unit circle $Im(z) = \sin\theta$

In addition $Im(z) = 0$ if z is real and also if the distribution of charges used is symmetrical as to OX, $Im\ W(z) = 0$, if z is real.

Therefore,

$$Im\ (z - W(z)) = 0 \tag{1}$$

is the equation of the circles as well as those parts of the axis of reals which connect the circles.

Suppose now that the charges are expressible by Fourier series in θ. By symmetry the terms in $\cos m\theta$ and $\sin 2m\theta$ drop out so that only sines of odd multiples of θ remain. Thus one may write

$$\sigma = -\frac{1}{4\pi}\sum_{s=1}^{\infty} a_{2s-1}\sin\ (2s-1)\theta$$

By a process exactly similar to the one used in Problem I it is found that (1) is equivalent to

$$\sin\theta_0 = -\sum_{s=1}^{\infty}\frac{a_{2s-1}}{2(2s-1)}\sin(2s-1)\theta_0 - \sum_{n=-\infty}^{n=+\infty}{}'\sum_{s=1}^{\infty}\frac{a_{2s-1}}{2(2s-1)}r_n^{-2s+1}\sin(2s-1)\theta \tag{2}$$

It is also shown without difficulty that

$$\sum_{n=-\infty}^{n=+\infty}{}'(z-nh)^{-2p+1} = -2\sum_{s=1}^{\infty}A_{2s+2p-2}z^{2s-1}\binom{2s+2p-3}{2p-2}h^{-2s-2p+2} \tag{3}$$

whence on reversing the order of summation in (2)

$$\sin\theta_0 = -\sum_{s=1}^{\infty}\frac{a_{2s-1}\sin(2s-1)\theta_0}{2(2s-1)} - \sum_{p,s=1}^{\infty}\frac{a_{2s-1}A_{2s+2p-2}}{2s-1}h^{-2s-2p+2}\binom{2s+2p-3}{2s-2}\sin(2p-1)\theta_0$$

Consequently since this equation is to be satisfied identically if $p > 1$

$$\frac{a_{2p-1}}{2p-1} + \sum_{s=1,2,..}^{\infty}\frac{2a_{2s-1}}{2s-1}A_{2s+2p-2}\binom{2s+2p-3}{2s-2}h^{-2s-2p+2} = 0 \tag{4}$$

and also setting $p = 1$

$$1 = -\frac{a_1}{2} - \sum_{s=1}^{\infty}\frac{a_{2s-1}}{2s-1}A_{2s}\,h^{-2s}\binom{2s-1}{2s-2} \tag{5}$$

It is convenient to define b_{2s-1} by

$$b_{2s-1} = \frac{a_{2s-1}}{2s-1} \tag{6}$$

which changes (4), (5) into

$$b_{2s-1} + \sum_{s_1=1}^{\infty} 2A_{2s_1+2s-2} \binom{2s+2s_1-3}{2s_1-2} h^{-2s-2s_1+2} b_{2s_1-1} \tag{7}$$

and

$$1 + \frac{b_1}{2} + \sum_{s_1=1}^{\infty} A_{2s_1} h^{-2s_1} \binom{2s_1-1}{2s_1-2} b_{2s_1-1} = 0 \tag{8}$$

Equations (7) are suited for the determination of b_{2s-1} in terms of b_1 When this determination has been effected equation (8) may serve to determine b_1 itself. Equation (7) may be rewritten as

$$\frac{b_{2s+1}}{b_1} = -2A_{2s+2} h^{-2s-2} - \sum_{s_1=1}^{\infty} \frac{b_{2s_1+1}}{b_1} 2A_{2s+2s_1+2} \binom{2s+2s_1+1}{2s_1} h^{-(2s+2s_1+2)} \tag{9}$$

If now it be true that

$$\frac{b_{2s+1}}{b_1}$$

may be expanded in a power series in $\frac{1}{h}$ then the coefficients in this power series may be obtained by collecting equal powers of h in the symbolical expression

$$\frac{b_{2s+1}}{b_1} = -2A_{2s+2} h^{-(2s+2)} + \sum_{s_1=1}^{\infty} 2^2 A_{2s_1+2} A_{2s+2s_1+2} \binom{2s+2s_1+1}{2s_1} h^{-(2s+4s_1+4)}$$
$$- \sum_{s_1,s_2=1}^{\infty} 2^3 A_{2s_2+2} A_{2s_1+2s_2+2} A_{2s+2s_1+2} \binom{2s+2s_1-1}{2s_1} \binom{2s_1+2s_2+1}{2s_2} h^{-(2s+4s_1+4s_2+6)}$$
$$+ \ldots \tag{10}$$

Substituting this into (8) b_1 may be determined.

It remains to justify the supposition that b_{2s+1}/b_1 may be expanded in a power series in $\frac{1}{h}$. This may be carried through by a method similar to that of Problem I.

Expanding in detail it is found that $-\frac{b_{2s+1}}{b_1} \frac{h^{2s+2}}{2}$ has the form stated in the text under *Discussion of Problem II*.

Appendix III

Solution of Problem III

Statement of Problem

For the same system of circles as in Problems I and II find such a field **H** that div **H** = 0, that **H** is tangential to the circles, that the line

integral of **H** around any closed curve should vanish and that at an infinite distance to either side of the circles the field should be uniform and perpendicular to OX.

The solution of this problem is very analogous to that of Problem II.

If one should find such a $W(z)$ that on the circle $|z| = 1$, $R(W(z)) = \cos\theta$, then $R(z - W(z)) = 0$ on the circle and also on OY provided $R(W(z)) = 0$ when $R(z) = 0$. Under these conditions

$$Im\,(z - W(z))$$

may be taken as the potential.

It will be attempted to find the $W(z)$ by finding a distribution of charge which will give a potential $R(W(z))$ which on the circle $|z| = 1$ degenerates into $\cos\theta$.

Imagine the distribution of charge to be a surface distribution of density

$$\sigma = -\frac{1}{4\pi}(a_1\cos\theta + a_3\cos 3\theta + \quad . \quad . \quad . \quad)$$

Then the *nth* section contributes to the potential at $\rho = 1$, $\theta = \theta_0$ the amount

$$\frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{(nh - \cos\theta + \cos\theta_0)^2 + (\sin\theta - \sin\theta_0)^2} \sum_{s=1}^{\infty} a_{2s-1}\cos(2s-1)\,\theta d\theta$$

Summing this as to n adding the contribution to the potential of the charges on the circle $|z| = 1$ itself and equating the result to $\cos\theta_0$ it is found that

$$\cos\theta = -\sum_{s=1}^{\infty}\frac{a_{2s-1}}{2(2s-1)}\cos(2s-1)\theta_0 - \sum_{n=-\infty}^{n=+\infty}{}' \sum_{s=1}^{\infty}\frac{a_{2s-1}}{2(2s-1)} r_n^{-2s+1}\cos(2s-1)\,\theta_n$$

in the previous notation.

By (3) of Problem II

$$\sum_{n=-\infty}^{n=+\infty}{}' r_n^{-2p+1}\cos(2p-1)\theta_n = -2\sum_{s=1}^{\infty} A_{2s+2p-2}\cos(2s-1)\theta \binom{2s+2p-3}{2p-2} h^{-2s-2p+2}$$

whence

$$\cos\theta_0 = -\sum_{s=1}^{\infty}\frac{a_{2s-1}}{2(2s-1)}\cos(2s-1)\theta_0 + \sum_{s,p=1}^{\infty}\frac{a_{2s-1}}{2s-1} A_{2s+2p-2}\binom{2s+2p-3}{2s-2} h^{-2s-2p+2}\cos(2p-1)\theta_0$$

the operations with series being justifiable as before.

Hence

$$\left\{\begin{aligned} \frac{a_{2p-1}}{2p-1} &= \sum_{s=1}^{\infty} 2a_{2s-1}(2s-1)^{-1}A_{2s+2p-2}\binom{2s+2p-3}{2s-2}h^{-2s-2p+2} \qquad \text{if } p>1 \\ \text{anh } 1 &= -\frac{a_1}{2} + \sum_{s=1}^{\infty} a_{2s-1}(2s-1)^{-1}A_{2s}\binom{2s-1}{2s-2}h^{-2s} \end{aligned}\right.$$

or letting

$$\beta_{2s-1} = a_{2s-1}/(2s-1)$$

$$\beta_{2s-1} = \sum_{s_1=1}^{\infty} 2A_{2s_1+2s-2}\binom{2s+2s_1-3}{2s_1-1}h^{-2s_1-2s+2}\beta_{2s_1-1}$$

$$1 = -\frac{\beta_1}{2} + \sum_{s_1=1}^{\infty} A_{2s_1}\binom{2s_1-1}{2s_1-2}h^{-2s_1}\beta_{2s_1-1}$$

These equations are very similar to equation (9) of Problem II and are solved in the same way. It is thus found that

$$\frac{h^{2s+2}}{2}\frac{\beta_{2s+1}}{\beta_1} = A_{2s+2} + 2A_4A_{2s+4}\binom{2s+3}{2}h^{-6} + 2A_6A_{2s+6}\binom{2s+5}{4}h^{-10} + 2^2A_4A_6A_{2s+4}\binom{2s+3}{2}\binom{5}{2}h^{-12} + \quad . \quad . \quad .$$

The computations here given were made partly while the writer was employed at the Bureau of Standards, and partly while he was a Fellow of the National Research Council. It is a pleasure to acknowledge here indebtedness to both of these institutions.

DEPARTMENT OF TERRESTRIAL MAGNETISM,
CARNEGIE INSTITUTION OF WASHINGTON,
WASHINGTON, D. C.

Rate at which an Electric Arc is Displaced Sidewise by a Transverse Magnetic field.—This phenomenon is studied by forming an arc between the adjacent edges of two hollow cylinders of equal size, mounted side by side upon a shaft, and immersed in the radial magnetic field formed by two solenoids mounted upon the extremities of the same shaft, with currents flowing through them in opposite directions so as to annul the axial component of the field. In this field, therefore, the arc continually revolves around the edges of the cylinders, at a measurable rate which according to simple kinetic theory should be equal (when expressed in terms of the linear speed of the moving arc) to the product of the magnetic field, the potential gradient in the arc, and the mobilities of the positive and negative ions; it should, therefore, serve as an experimentally accessible measure of the product of these two mobilities, or of one of them provided the other can be determined *a priori*. The experiments were performed in air at atmospheric pressure

with electrodes of several metals and of carbon; the currents ranged up to 12 amp., the arc-lengths from 1 to 7 mm (which seem unfortunately short, since the regions near the electrodes must play relatively too great a part). The rate was found not quite, but fairly closely, proportional to the magnetic field; it also varies with the length of the arc, a behavior of which the theory gives no anticipation, but which is presumably connected with the anomalies near the electrodes and with the odd crooked shapes which the arc sometimes assumes. The question of dependence on potential gradient is not so easy to answer, since when this is varied the current and the temperature of the arc also vary, resulting in variations of rate of which the theory gives no account. But by assuming the theory to be valid under any particular set of circumstances and measuring the rate, the value of the product of the mobilities under those circumstances can be estimated. The first result of proceeding in this manner is, that the product of the mobilities is found to depend on the material of the anode, not on that of the cathode—which implies that the negative ions in the arc are electrons or negatively-charged gas-molecules, while some at least of the positive ions actually proceed from the metal of the anode. This is an important result in accord with conclusions from much other evidence. Next, when the negative ions are assumed as a hypothesis to be electrons, and the theoretical value for the mobility of these is inserted into the experimental value for the product of the mobilities, the deduced value for the mobility of the positive ions comes out in fair agreement with the idea that they are molecules of oxygen and nitrogen with an admixture of ions from the metal. Accurate tests are impossible to make since the calculation involves an assumption as to the temperature of the arc.—[H. Stolt, Upsala; Ann. d. Phys. *74*, pp. 80-104; 1924.]

KARL K. DARROW

Ranges of the Alpha-particles from UI and UII Deduced from Pleochroic haloes.—In a mineral called "Wolsendorfer fluorspar" admirable pleochroic haloes are seen which lend themselves very readily to exact measurement, for the alpha-particles bleach the mineral all along their range except at the very end where they produce a sharp dark ring. Beautiful microphotographs of systems of three or four such concentric rings, due to consecutive members of the uranium-radium series, are shown in the article (the diameters of the actual rings range between 15 and 35 μ). Rings due to eight members of the series are measured; for the six of which the ranges in air are accurately known, the ratio of the ranges in air and in fluorspar is sensibly constant; assuming it likewise constant for the remaining two (the rings of which are fused into one broadened circle), Gudden calculates 2.76 and 2.68 cm for the ranges in air characterizing UII and UI respectively; these values, when compared with the estimated half-periods of these elements, are not concordant with the Geiger-Nuttall relation between half-period and range.—[B. Gudden, Göttingen; ZS. f. Phys. *26*, pp. 110-116; 1924.]

KARL K. DARROW

A THEORY AND USE OF THE BUMSTEAD ELECTROSCOPE*

By Kenneth Cole

In 1806, Behrens[1] described a new electroscope in which a gold leaf suspended between two electrodes charged to equal and opposite potentials was used to indicate and—to a certain extent—measure small charges. The case of the instrument seems to have been a glass tumbler while a pile of eighty voltaic elements consisting of brass, tinfoil, and salted gilt paper was used to charge the electrodes. The only idea of the sensitivity attained is to be gotten from the author's statement that the charged body must be brought slowly towards the plate which topped the support for the leaf. Although this self-same instrument has been rediscovered, used, or modified many times since 1806,[2] it was not until the work of Bumstead and his contemporaries[3] that the arrangement became the accurate and reliable instrument that is now commonly known as the Bumstead electroscope. Its advantages are its comparatively high sensitivity, low electrostatic capacity, ruggedness and ease of construction. The simplicity of the electrical system invites an analytical treatment which is here undertaken.

Theory

By the method of images, Kuntz and Bayley[4] have shown that when an infinite linear charge of e per unit length is placed a distance a from one of two infinite parallel plates whose separation is l as shown in Fig. 1, the potential of any point referred to the axes indicated is

$$V_p = e \log_e \frac{\cosh\frac{y}{l}\pi - \cos\frac{x+a}{l}\pi}{\cosh\frac{y}{l}\pi - \cos\frac{x-a}{l}\pi} \qquad \text{(I)}$$

* See H. S. Read, Phys. Rev. *21*, p. 368, 1923.

[1] Behrens, Gilbert's Annalen, *23*, p. 24, 1806.

[2] Kleefeld, Gilberts Annalen, *34*, p. 203; 1810.
Bohnenberger, Annales de Chimie et de Physique, *16*, p. 91; 1821.
Becquerel, Annales de Chimie et de Physique, *46*, p. 276; 1831.
Hankel, Poggendorff's Annalen, *103*, p. 209;1858.
and others.

[3] Bumstead, Phil. Mag. *22*, p. 910; 1911.
Dadourian, Phys. Rev. *14*, p. 238; 1919.

[4] Kuntz and Bayley, Phys. Rev. *17*, p. 147, 1921.

Near the charge, the equipotential surfaces are circular cylinders to the second order of small quantities, when the charge is not near one of the plates. If the charge is replaced by a wire of radius r and charge e per unit length, then by putting $x=a$ and $y=r$, the potential of the wire is

$$V_w = e \log_e \frac{\cosh\frac{r}{l}\pi - \cos\frac{2a}{l}\pi}{\cosh\frac{r}{l}\pi - 1} \tag{II}$$

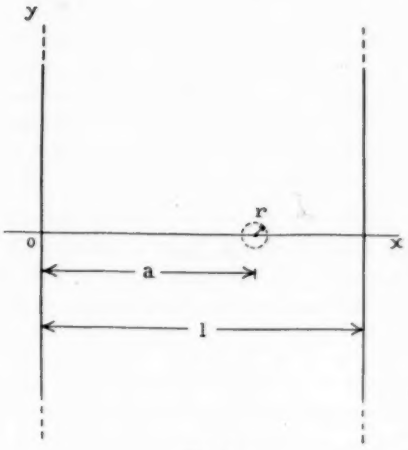

FIG. 1

and its capacity per unit length is

$$c = \frac{1}{\log_e \dfrac{\cosh\frac{r}{l}\pi - \cos\frac{2a}{l}\pi}{\cosh\frac{r}{l}\pi - 1}} \tag{III}$$

The x-component of the field at any point is

$$E_x = -\frac{\partial V_p}{\partial x} = -e\frac{\pi}{l}\left\{\frac{\sin\frac{x+a}{l}\pi}{\cosh\frac{y}{l}\pi - \cos\frac{x+a}{l}\pi} - \frac{\sin\frac{x-a}{l}\pi}{\cosh\frac{y}{l}\pi - \cos\frac{x-a}{l}\pi}\right\} \tag{IV}$$

gnoring the charges induced on the wire by its images and putting $x=a$ and $y=r$, the field at the wire due to its images is

$$E_1=-e\frac{\pi}{l}\frac{\sin\frac{2a}{l}\pi}{\cosh\frac{r}{l}\pi-\cos\frac{2a}{l}\pi} \qquad \text{(V)}$$

If d is now taken as the distance of the wire from the center position and $2s$ as the separation of the plates, Fig. 2, then

$$a=s+d \text{ and } l=2s$$

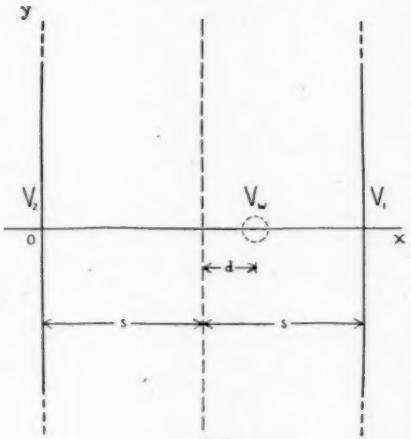

FIG. 2

Making these substitutions in (V) and expanding in series,

$$E_1=-e\frac{\pi}{2s}\left\{\frac{(-d\pi/s+\cdots\cdots)}{\left[1+\frac{1}{2!}\left(\frac{r\pi}{2s}\right)^2+\ldots\right]+\left[1-\frac{1}{2!}\left(\frac{d\pi}{s}\right)^2+\ldots\right]}\right\}$$

which becomes approximately

$$E_1=e\left(\frac{\pi}{2s}\right)^2 d \qquad \text{(VI)}$$

where r and d are small as compared with s. From (III) in a similar manner neglecting fourth and higher orders of small quantities,

$$c=\frac{1}{\log_e\left[1+\frac{16}{r^2}\left(\frac{s^2}{\pi^2}-\frac{d^2}{4}\right)\right]} \qquad \text{(VII)}$$

which becomes approximately

$$c=\frac{1}{2\log_e\frac{4s}{\pi r}}$$

if d and r are small as compared with s.

If the potential of the position of the wire be zero when d is zero and V_1 and V_2 (see Fig. 2) are other than zero, then the charge on a wire of length l is

$$q=cl(V_w-\frac{V_1-V_2}{2s}d)$$

Since the electrostatic force on the wire is

$$F=Eq$$

where (putting q/l for e in (VI) and neglecting the second order term in d).

$$E=cV_w\left(\frac{\pi}{2s}\right)^2 d-\frac{V_1-V_2}{2s}$$

then, calling the mechanical restoring force per unit displacement k, for equilibrium,

$$kd=c\,l\left[cV_w^2\left(\frac{\pi}{2s}\right)^2 d+\left(\frac{V_1-V_2}{2s}\right)^2 d-V_w\frac{V_1-V_2}{2s}\right] \qquad \text{(VIII)}$$

neglecting second order terms in d. This may be written

$$d=-\frac{c\,l\,V_w\frac{V_1-V_2}{2s}}{k-c\,l\left[cV_w^2\left(\frac{\pi}{2s}\right)^2+\left(\frac{V_1-V_2}{2s}\right)^2\right]} \qquad \text{(IX)}$$

If V_w is small as compared with V_1-V_2, the sensitivity is

$$S_b=\left|\frac{d}{V_w}\right|=\frac{\left|c\,l\frac{V_1-V_2}{2s}\right|}{k-c\,l\left(\frac{V_1-V_2}{2s}\right)^2} \qquad \text{(X)}$$

which is the equation given by Swann.[5] But if V_w is large as compared with V_1-V_2, then the sensitivity is

$$S_c=\left|\frac{d}{V_1-V_2}\right|=\frac{\left|cl\frac{V_w}{2s}\right|}{k-c^2l\left(\frac{V_w}{2s}\right)^2\pi^2} \qquad \text{(XI)}$$

[5] W. J. G. Swann, Phys. Rev. *23*, p. 779; 1924.

This use corresponds to the usual use of the quadrant electrometer in which the needle is kept at a constant potential, with one pair of quadrants grounded and the other pair charged with the charge or potential to be measured. The electrostatic control in (X) is due to the charge induced on the wire by the plate; and in (XI) it is supplied by the charges induced on the plates,—that is, it is the force exerted on the wire by its images.

EXPERIMENTAL

A gold leaf electroscope 25 cm high and 25 cm in diameter with plates 12 cm by 15 cm moving parallel to each other by micrometer screws was used to test the theory. The image of the gold leaf was projected on a translucent screen with a magnification of 40X with a camera lens. The illumination was supplied by a concentrated filament tungsten lamp, but in order that troublesome convection currents within the instrument[6] should be small, it was necessary that the light actually used in forming the image on the retina of the eye should be as large a part of the radiant energy entering the electroscope as possible. To this end, a water cell and a cell containing a .32 N solution of anhydrous $CuCl_2$ in absolute alcohol, were used as filters, the latter transmitting from .515μ to .590μ, which is about the region of maximum visual sensitivity. In order to obtain a clean cut image, a quartz fiber was often pressed onto the end of the leaf.

On several occasions the amber insulation of the leaf support became charged by friction and the leaf wrapped itself very tightly around the amber. If the amber were then exposed to x-rays for a short time, the leaf could be disengaged from it without damage.

The theory was developed above for a round wire, uniformly charged, and of infinite extent, moving parallel to itself. Since the experimental work was confined to narrow leaves from .3 mm to 2.0 mm wide and from 4.0 cm to 7.5 cm long, swinging about their upper ends, experiment and theory cannot be expected to agree with any high degree of precision. To find accurately and take into account the actual distribution of charge on a wire or leaf swinging about a fixed point would require a three dimensional analysis that has not been attempted.

In the first place, it was found for either connection of the electroscope that the sensitivity was directly proportional to the length of leaf. This was not to be expected from equations (X) and (XI) if the two terms in the denominators are comparable as they were in the

[6] Farwell, H. W. Am. Jr. Sci. *37*, p. 319; 1914.

experimental work, and if d is taken as the displacement of the lower end of the leaf. If, however, we call this displacement p and the mass of the leaf per unit length m, then the mechanical restoring force is mgp and $k=mg$. If we now assume that the induced charge is heavily concentrated on the end of the leaf, proportional to p and independent of l, and—as the theory indicates—consists of two parts, one proportional to $\left(\frac{V_w}{2s}\right)^2$ and the other proportional to $\left(\frac{V_1-V_2}{2s}\right)^2$ then we would have instead of (VIII)

$$mgp=\alpha\left(\frac{V_w}{2s}\right)^2 p+\beta\left(\frac{V_1-V_2}{2s}\right)^2 p-clV_w\frac{V_1-V_2}{2s}$$

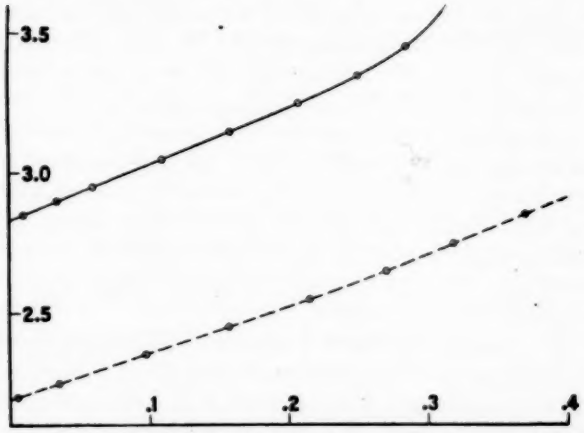

FIG. 3. *Solid curve: abscissae are $1/S_b$; ordinates are s; dotted curve: abscissae are $1/S_c$; ordinates are s; s is in cm; S_b and S_c are cm deflection $\times 40$ per volt. $V_1-V_2=V_w=234$ volts. Gold leaf 5.9 cm long, .5 mm wide.*

and instead of (X) and (XI)

$$S_b=\left|\frac{p}{V_w}\right|=\frac{\left|cl\frac{V_1-V_2}{2s}\right|}{mg-\beta\left(\frac{V_1-V_2}{2s}\right)^2} \qquad \text{(XII)}$$

and

$$S_c=\left|\frac{p}{V_1-V_2}\right|=\frac{\left|cl\frac{V_w}{2s}\right|}{mg-\alpha\left(\frac{V_w}{2s}\right)^2}. \qquad \text{(XIII)}$$

Thus on the basis of the assumption given above, although it cannot be more than a very rough approximation, we have both sensitivities proportional to the length of the leaf.

For almost any shape and symmetrical arrangement of the electrodes, the deflections up to $s/20$ are found to be proportional to the potential being measured. This range can be increased by using plates set at an angle, which at the same time makes it impossible for the leaf to touch the plate. Large and small plates and wire electrodes, both parallel and inclined at angles up to 30° from the vertical in either direction, give the relation shown in Fig. 3 between the sensitivity for both uses and the plate separation. The linear relation for high sensitivity is not predicted by the theory to hold over as long a range as it does. When the plate distance remains fixed, equations (XII) and (XIII) may be rewritten

$$\frac{\frac{V_w}{2s}}{S_c} = -\frac{\alpha}{l}\left(\frac{V_w}{2s}\right)^2 + \frac{mg}{cl},$$

$$\frac{\frac{V_1 - V_2}{2s}}{S_b} = -\frac{\beta}{l}\left(\frac{V_1 - V_2}{2s}\right)^2 + \frac{mg}{cl}$$

and when plotted as indicated in Fig. 4 should give a straight line in which the intercept on the axis of ordinates will be the same for both uses. The ratio of $\frac{V_w}{V_1 - V_2}$ for $S_b = S_c$, with the same plate distance and leaf, varies up to a maximum of 1.4 at the highest sensitivities.

In the Bumstead connection, the two plates must be kept accurately at equal and opposite potentials, since any differential change causes a serious change in the zero position of the leaf. This difficulty has been avoided to a certain extent by shunting the battery with a pair of high resistances and grounding their center connection. When however the whole battery is connected to the leaf and to ground, with one plate grounded and the other connected to the potential to be measured, the only effect of a change of voltage is to change the sensitivity approximately the same as it is changed for the Bumstead connection.

In the present instrument, the convection currents are troublesome and the gold leaf is not sufficiently flexible. The highest sensitivity that it has been possible to obtain with the Bumstead connection is 38,000 mm per volt with a magnification of 40X. The deflections were linear with the applied potential but the zero varied somewhat. A

sensitivity of 10,000 mm per volt at the same magnification has been used with the quadrant electrometer connection, but no attempt has been made to go higher although the zero was remarkably constant. The sensitivity is easily controlled by changing the leaf potential, leaving the plates fixed in position and leveling the instrument to secure equal deflections in both directions.

No work has been done on the use of this connection for the measurement of charge. There is the disadvantage that the capacity of the plate cannot be made as small as that of the leaf. It is worthy of note

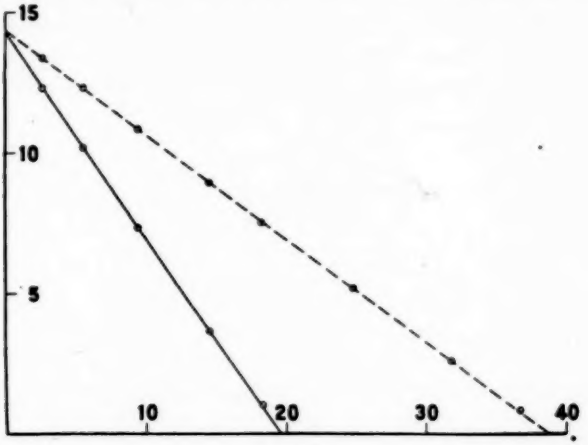

FIG. 4. *Solid curve: abscissae are* $\left(\frac{V_1-V_2}{2s}\right)^2 \times 10^8$; *ordinates are* $\frac{V_1-V_2}{2s} \cdot \frac{1}{S_b} \times 10^8$ *dotted curve: abscissae are* $\left(\frac{V_w}{2s}\right)^2 \times 10^8$; *ordinates are* $\frac{V_w}{2s} \cdot \frac{1}{S_a} \times 10^8$; V_w, V_1, *and* V_2 *are in e.s.u.,* $s=3.23$ *cm,* S_b *and* S_a *are the deflections of the tip of the leaf in cm per e.s.u. Gold leaf 6.65 cm long, .45 mm wide.*

that the variation of the charge induced on the plate is negligible for small deflections as can easily be shown. The total charge induced on one plate per unit length is

$$e' = 2\int_0^\infty \sigma dy = \frac{1}{2\pi}\int_0^\infty (-E_x)_{x=0} dy$$

where σ is the surface density of charge. Then substituting in the above, values obtained from equation (IV) we obtain approximately,

$$e' = -\frac{e}{2}\left(1-\frac{d}{s}\right)$$

so that if d is small in comparison with s, the capacity of the plate will not be sensibly affected by the movement of the leaf and the deflection should be directly proportional to the charge placed on the plate.

An electroscope is now being constructed that will use a conducting quartz fiber and can be evacuated. It is hoped that convection currents can be lessened and a higher magnification used, making it easy to obtain and use high sensitivities.

The author wishes to express his appreciation of the assistance and encouragement of Professor F. K. Richtmyer and Professor E. H. Kennard.

ROCKEFELLER HALL, CORNELL UNIVERSITY,
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Prevention of Blindness.—"The greatest possibilities for the elimination of the unnecessary eye hazards of industrial occupations lie in education—education of the state as to its moral and economic obligations; education of the employer as to the desirability, from his own point of view, of supplying adequate protection for the eyes of his employes; and education of the employe as to the necessity of using that protection when it is provided."

This is the conclusion of the National Committee for the Prevention of Blindness, New York City, which has recently made public its findings in a two years' study of the eye hazards in industrial occupations which extended into practically every industry and every State in the Union.

"While it is patent that safety cannot be legislated into an industrial plant," says the report of the Committee, a document of some 250 pages, "nevertheless it is futile to expect certain types of employers to give serious attention to the prevention of accidents, to the provision of proper lighting facilities, and to sanitation within their property if the state itself gives little attention to these matters. Some states do not yet require even the reporting of industrial accidents. Others require such reports, but make no effort to set up safety standards of lighting and sanitation codes for the guidance or observance of their industries notwithstanding the fact that it is now a simple matter for any state to obtain from the more progressive industrial states, from national trade associations and technical societies, from the United States Bureau of Standards, or from the American Engineering Standards Committee drafts of safety codes prepared by experts representing all interested parties and approved as national standards.

"The state's opportunities in this direction will not be fulfilled until legislators and administrative officials of every state recognize that the prevention of accidents is a moral and economic obligation of the state as well as of the individual employer. The State machinery best suited for the formulation and the administration of industrial

safety regulations appears to be an industrial commission with full power to make and administer rules which may be varied from time to time to conform to changes in industrial conditions, rather than a code of laws which can be changed only by a legislature which meets once in two years and which is often influenced by political expediency."

Under the heading "It is Cheaper to Prevent Accidents than to Pay for Them" the National Committee says:

"The report growing out of the survey of waste in industry, which was conducted by the Federated American Engineering Societies under the direction of Herbert Hoover, shows an estimated loss of 158,000,000 days as the result of non-fatal accidents occurring in industry in one year. On the extremely conservative estimate of $1.50 a day as the average compensation paid to workmen for time lost due to accidents, there is a loss to employers of $237,000,000 a year as the result of non-fatal accidents alone. It is conservatively estimated, also, that eye accidents constitute 10 per cent of all non-fatal industrial accidents. On the basis of these estimates it appears that the loss to industry through eye accidents is in excess of $23,000,000 yearly. No employer or industrial manager who is aware of these facts will question the assertion that it is cheaper to prevent eye accidents than to pay for them.

The report contains many practical suggestions for eye protection. [Report of the National Committee for Prevention of Blindness, 130 E. 22nd St., New York City.]

Spectra of Neutral Helium and of Ionized Helium in the Extreme Ultraviolet.—In this extension of Lyman's already classical researches in the extreme ultraviolet, he announces that he has discovered two members of the highest-frequency series in the spectrum of He+; has confirmed and established the position of the highest-frequency series in the spectrum of He, locating seven lines of it to within 0.1 or 0.2A; has discovered a continuous band extending towards higher frequencies from the limit of the latter series; and in addition has fortified the experimental evidence that conspicuous lines at 600.3A and 591.6A belong really to helium, although the transitions to which they correspond (possibly the former does not exactly correspond to any) are "prohibited" by the selection-principle except in abnormal conditions. The article abounds in details of the experimentation and of the precautions which must be taken before a photographed line can be accepted as due to helium, or even as real; in particular it is indispensable to photograph repeatedly with different gratings ruled by different engines. There is a beautiful photograph of the highest-frequency series of neutral He, which is probably the highest-frequency series emitted by any neutral atom.—[T. Lyman, Harvard; Astroph. Jl. *60*, pp. 59-84; 1924.]

KARL K. DARROW

THE EFFECT OF TRANSVERSE JOINTS ON THE MAGNETIC INDUCTION IN NICKEL

By S. R. Williams

ABSTRACT

Maurain found that the outer layers of nickel were more magnetic than the inner ones. N. H. Williams said this must lead to an increased induction in a nickel rod if it were cut into a number of sections. He found that transverse sections did increase the induction for strong fields but lowered it for weak. This paper shows that because of this effect a bundle of nickel plates will orient itself in a magnetic field differently for weak fields than for strong. A possible explanation lies in the readiness which the elementary magnets of nickel have to align themselves with a magnetizing force, particularly those in the outer layers which are not affected so much by the mutual directive forces of the elementary magnets. The increase in permeability due to mechanically sectioning nickel bars seems to indicate that somewhat analogous conditions are produced when the permeability of a substance is increased by heat treatment.

INTRODUCTION

In studying the magnetic properties of thin films of nickel and iron, which were deposited electrolytically on cathodes of various metals, Maurain[1] found that a certain definite thickness must be deposited before their magnetic properties manifested themselves. In the case of nickel this was about 200$\mu\mu$, while for iron the thickness was about 80$\mu\mu$. These thicknesses are greater than molecular dimensions, hence these experiments constitute an indication that magnetism may not be a molecular phenomenon, after all, but is a property of a crystal or some other large group.

The work of Maurain led N. H. Williams[2] to investigate the effect of transverse joints on the magnetic induction in nickel bars. "For," as he argued, "if the surface layers of nickel are more strongly magnetized than the mass of nickel it would seem that, in this metal, the transverse joints would tend to increase the magnetic induction rather than decrease it as in iron."[3]

In the effect of transverse joints on magnetic induction some of the factors which enter into the problem should be kept clearly in mind. First, the sections introduce air-gaps which increase the reluctance of the magnetic circuit in the specimen tested. Second, the Villari effect, (effect of pressure on magnetization) is always present and third, if

[1] Maurain, Jour. de Phys., *1*, p. 151, 1902.
[2] Williams, Phys. Rev., *33*, p. 60; 1911.
[3] Ewing and Low, Phil. Mag., *26*, p. 274; 1888.

the magnetic induction is augmented or diminished by transverse sectioning the greater the number of sections introduced the greater will be the effect.

The results of Williams disclose the fact that for field strengths up to 25-30 gauss, transverse joints in nickel decrease the magnetic induction, while for fields greater than this there is an increase. Hence, it may be argued that if a bundle of nickel plates was built up in the form of a cube and the same suspended by a fiber in a magnetic field it would tend to turn with sections parallel to the field for small magnetizing forces, and normal to magnetizing force for strong magnetization. It is the purpose of this paper to study the behavior of such a group of plates in the magnetic field, when the direction of the field initially makes an angle of 45 degrees with the sections of the group. See Fig. 3.

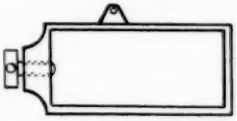

FIG. 1. *Stirrup to hold nickel plates.*

APPARATUS

The block of plates was built up from sheets of nickel about 0.1125 cm thick, each plate measuring one cm on a side. The surfaces were made good enough mirrors to see in them bright objects, located some distance from the mirrors. The polishing was done on a fine hone which had been selected for its plane surface. The condition of the surfaces seemed to play an important role in this work also. Doubtless the degree of reluctance introduced into the magnetic circuit is the chief factor affected by the surfaces. For holding the plates in position and keeping them together a brass stirrup was used as shown in Fig. 1. It was made to hold a group of plates whose length normal to sections was greater than one centimeter, if so desired. When a pile of plates was used so that a cube was formed the extra space was filled with brass plates. By means of a screw at one end the pressure was varied. An approximate determination of the pressure applied was obtained by applying a known torque to the head of the screw. The pitch of the screw was 0.0794 cm. The torque was measured by applying various forces at the end of a lever, 20 cm long, one end of which fitted into a hole in the head of the screw. The deflection of the stirrup with the nickel plates was read by means of a telescope, T, and scale, S_1, Fig. 2. The centi-

meter scale was bent in a circle whose radius was such that one centimeter on the scale equaled one degree of rotation. See Fig. 3. It was found that there was a small magnetic moment due to the brass stirrup and brass plates, but this was so small as to be negligible in comparison with deflections when the nickel plates were in position.

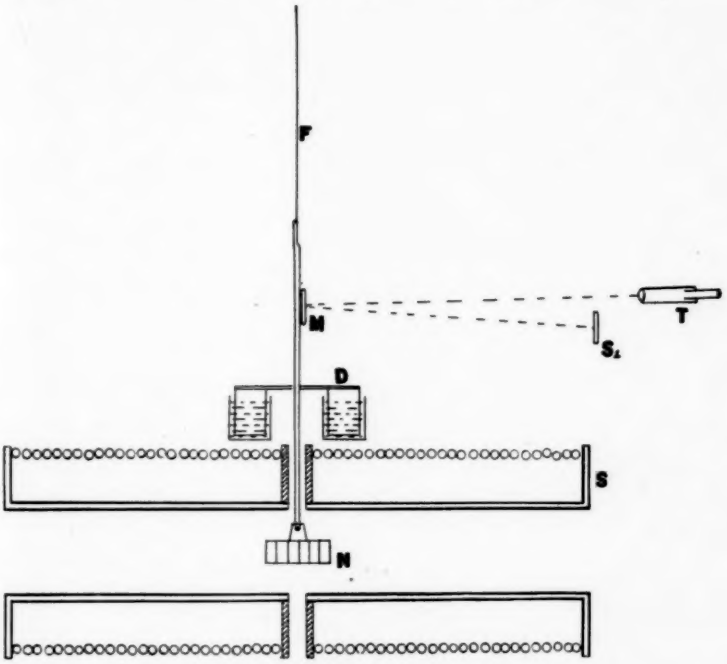

FIG. 2. *Vertical section of apparatus*

A solenoid, S, 32.90 cm long and with an inner free diameter of 7.00 cm, served as a means for producing a uniform field. It was wound in two sections of eight layers, a brass cylinder serving as the bobbin. Brass plates, 10×10 cm, formed the ends to the spool on which the wire was wound (see Fig. 2.). A wooden annular strip was put around the brass cylinder at the middle to serve as a separator between the two sections of the coil. Through this wooden separator a small hole was bored normal to the axis of the coil to allow the shaft to which the mirror was attached to pass from the fiber, F, outside to the stirrup and plates inside. Fig. 2. To steady the deflections a damper, D, of mica vanes was swung in an annular dish of oil. The brass spool, on

which the solenoid was wound, was slotted its entire length including the end pieces so that when an alternating current was sent through the coil for demagnetizing purposes a minimum amount of eddy currents would be set up in the brass. The constant of the solenoid as determined by calculation and confirmed experimentally was 44.7 gauss per ampere.

In order to minimize the effect of the horizontal component of the earth's magnetic field, two tangent galvanometer coils, C_1 and C_2, Fig. 3, were set up to form a Helmholtz coil and so placed that the axes of the

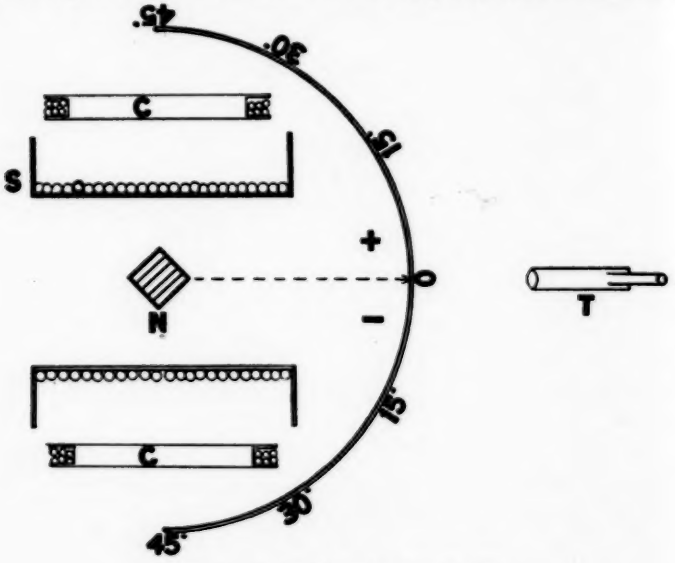

FIG. 3. *Horizontal section of apparatus.*

coils were normal to the main solenoid and parallel to the magnetic meridian. A current of 0.375 amperes through this coil just annulled the horizontal component when oppositely directed. After each set of readings the nickel plates were demagnetized in a decreasing alternating magnetic field in this space which was free of extraneous magnetic fields.

A liquid rheostat[4] was used to control the magnetizing current. The range was from about five up to five hundred ohms, depending upon the acidity of the tap-water used. It gave a very steady control of the current in the solenoid from small values upward.

[4] Williams, School Sci. and Math., *12*, p. 489; 1912.

RESULTS

With the use of nine plates of nickel whose combined thickness normal to the sections was 0.978 cm, the plates always turned so that the sections were parallel to the field, and this behavior persisted no matter what pressure was applied to them. The maximum pressure for this set of plates was 40 kg. Curve *b*, Fig. 4, shows the characteristic

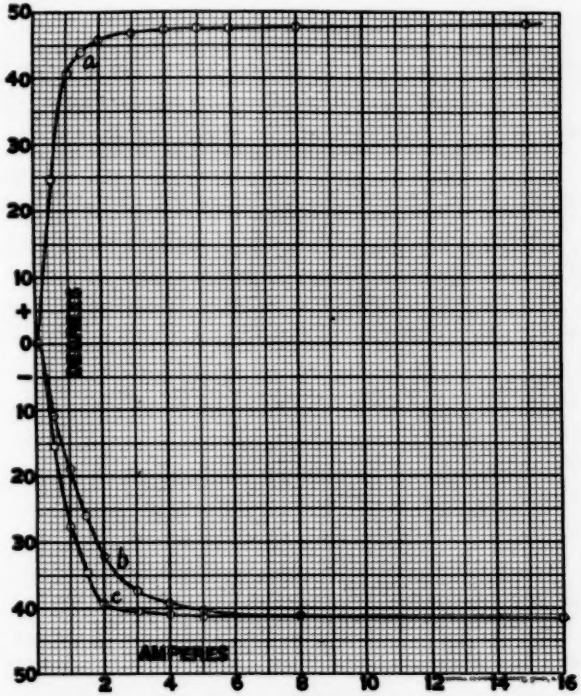

FIG. 4. *Deflections of piles of nickel plates, 1 cm square.*
Curve a: 10 plates, total thickness 1.148 cm
Curve b: 9 plates, total thickness 1.005 cm
Curve c: 9 plates, 1.224 cm, separated by paper; pressure 75 kg.

(not actual) behavior of the plates for this dimension. It is plain why this turning occurred, for not only is the length of the block of plates greatest parallel to the sections but also the air gaps between the plates aided in the same direction. The increase in induction due to the sections as found by Williams and that due to the pressure were not sufficient to overcome the other two effects.

Again another set of nine plates was made up whose total length normal to the sections was 1.005 cm. This length was now a trifle longer than that parallel to the sections. If it had been a solid cube of homogeneous nickel it would have turned with the 1.005 cm dimension parallel to the magnetizing force, but this bundle of plates still persisted in turning as the preceding one. The actual readings are shown in curve *b*, Fig. 4. Magnetic reluctance was still too great.

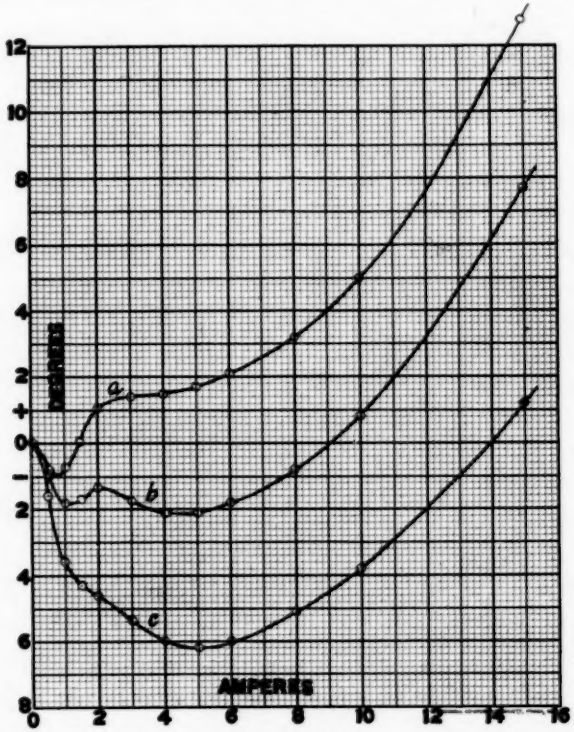

FIG. 5. *Deflection of a pile of 9 plates of thickness 1.031 cm under pressures of (a) 60 kg, (b) 36 kg, (c) 12 kg.*

Another selection of nine plates whose combined thickness was 1.031 cm gave different results. The various positions which the plates took as the field was increased are shown in *a*, *b* and *c*, Fig. 5, when pressures of 60, 36, and 12 kg respectively were applied. Here it appears that the bundle of plates behaves for small fields like a diamagnetic substance and like a paramagnetic for strong. This is in line with what

Williams found, viz., a decrease in induction for weak fields and an increase for strong. The increasing pressure seems also to be wiping out the effect of the air-gaps, causing the *apparent* diamagnetic properties to disappear.

In order to study the effect of changing the plates while keeping the length normal to the sections the same, a similar group was built up

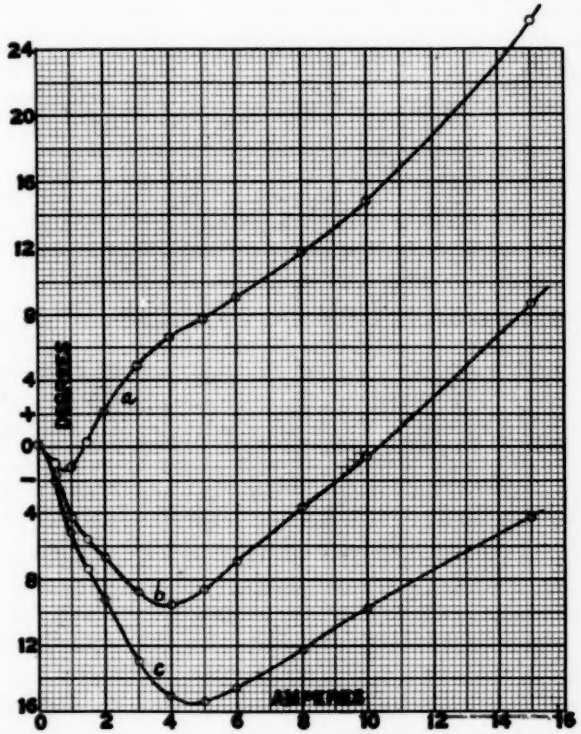

FIG. 6. *Deflection of a second pile of 9 plates, of thickness 1.031 cm, under pressures of (a) 75 kg, (b) 36 kg, (c) 12 kg.*

having the same length, viz., 1.031 cm. When one reflects that it would be practically impossible to duplicate the surface condition on each plate, it is not surprising to find no quantitative agreement between the results for corresponding pressures in the two cases. There is, however, sufficient qualitative agreement to show that there are similar conditions, barring the surface ones, which hold in the different nickel plates. The results for this second set of plates having a total

thickness of 1.031 cm are shown in *a*, *b*, and *c*, Fig. 6, with pressures of 75, 36 and 12 kg respectively.

In order to increase the effect of the air-gaps without changing the pressure the nine plates just used were separated from their neighbors by thin pieces of paper one centimeter square. These were then put under a pressure as high as 75 kg. The total thickness including the paper separators was 1.224 cm. The length normal to these sections was then considerably greater than parallel and also greater than for the plates without the paper, yet no matter what pressure was applied the group of plates always turned with sections parallel to the field. This is shown in *c*, Fig. 4.

When ten plates were grouped together without paper separators and their combined thickness was 1.148, their rotation in the magnetic field was always with the sections tending to become normal to the applied field. This is typified by curve *a*, Fig. 4. The reason is obvious.

In a similar fashion as for the nickel plates various groups of soft Swedish iron plates were built up and their behavior observed. These plates were also one centimeter square and were, after grinding, about 0.168 cm in thickness. No combination of iron plates was found which did not behave either as shown in curve *a* or curve *c* in Fig. 4. They were not, as were the nickel plates, first one thing for weak fields and another for strong. This agrees with Ewing and Low's work.

Steel magnets with laminations *parallel* to the axes of the magnets have been the subject of a number of interesting studies[5] all of which go to show that such laminations and separations of the sections produce a stronger magnet than a solid bar.

In Fig. 7 is shown the way in which another group of nickel plates, 1.034 cm in length, behaved when different initial settings were given to the stirrup holding the plates, the pressure remaining the same throughout. Curve *b* is the normal condition when the diagonal plane of the cube intersected the scale at zero or the diagonal plane of the cube coincided with the axis of the solenoid, Fig. 3, no external field being applied. Curve *a* is when the diagonal plane initially intersected the scale at $+10°$ and curve *c* the corresponding position at $-10°$. This group of plates was a trifle longer than those measuring 1.031 cm. As a result the initial turning toward negative values does not appear, but otherwise curve *b*, Fig. 7, is of the type of *a* Fig. 5.

[5] Scoresby, "Magnetical Investigations," 1839.
Jamin, Compt Rend, *75*, *76*, *77* and *81*; 1872-1875.
Brown, Proc. Roy-Soc - Dublin, *15*, p. 137; 1916.

DISCUSSION OF RESULTS

These results are a complete confirmation of Williams' experiments and may be interpreted by the assumption that above a certain critical field strength the surfaces of narrow slits cut in nickel, normal to the magnetizing force, are more susceptible than are the interior layers. This seems to be reasonable since the surface layers are not affected so much by the mutual directive forces of elementary magnets.

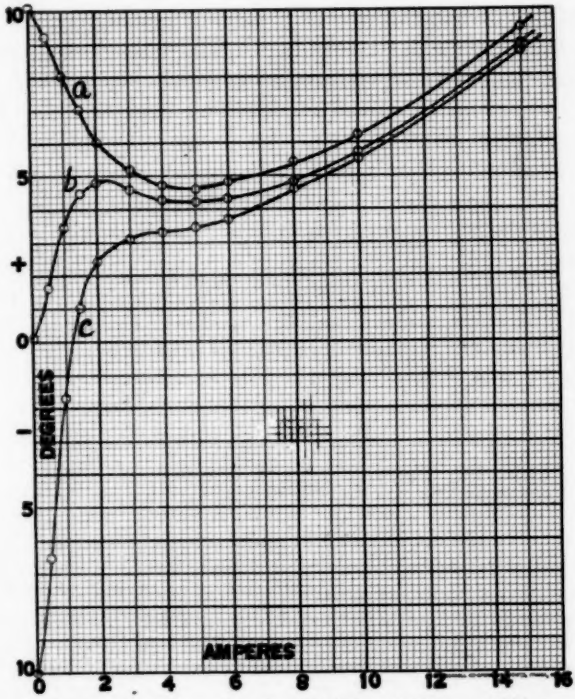

FIG. 7. *Deflections of a pile of 9 plates, of thickness 1.034 cm*
(a) Diagonal set at + 10° (b) diagonal set at zero (c) diagonal set at − 10°

In the foregoing experiments the reversal of direction for weak fields indicates that the reluctance of the air-gaps predominates, but as the applied field grows the susceptibility of the outer layers rapidly goes up and soon overcomes the reluctance of the air-gaps. If there is a difference in induction between the cut and the uncut bar it must be due to air-gaps and surface layers of the sections. This leads to the suggestion that if one could build up a mass of nickel composed wholly

of surface layers and air-gaps, reduced to a minimum, it ought to form a specimen of very high permeability. The Western Electric Company[6] have recently developed an alloy, (permalloy) of nickel and iron which has an unusually high permeability, especially at low field strengths. May they not have separated the nickel into a great many small aggregates thus giving almost an infinite number of surface layers and the air-gaps replaced by thin deposits of iron, producing by heat treatment what has here been accomplished by mechanical means? Such a point of view might lead to an explanation of some of the magnetic properties which the alloy possesses when treated mechanically and thermally. It might also lend significance to the fact that the distance between atoms seems to be the most important factor in the magnitude of the magnetic properties of ferromagnetic substances. This fact is brought to light repeatedly in the work of Ross[7] and others[8] on the magnetic properties of the Heusler alloys and is confirmed by the work of Hull[9] in the x-ray analysis of ferromagnetic substances.

This investigation was carried out in the Norman Bridge Laboratory of Physics. The author is greatly indebted to the Director, Professor R. A. Millikan, for the equipment necessary to carry on the problem and for many valuable suggestions while the work was going on.

PHYSICAL LABORATORY
AMHERST COLLEGE,
AMHERST, MASS.

Positive Ions from Warm Salts.—The salt was spread over a spiral platinum filament mounted inside a narrow glass tube which projected with its open end into the main tube; the ions were accelerated by a small voltage (2-8 volts) and deflected in a magnetic field to a disc electrode mounted obliquely to the open end of the narrow tube, the quantity measured being the magnetic field strength resulting in maximum current to the obliquely-placed electrode. It seems astonishing that any quantitative data should be obtained with so excessively simple a device, especially since Volmer added 1 volt to his applied P.D. to allow for the initial velocities of the ions, without however saying anything about contact P.D. However the measured values of e/m agree very well with the values to be expected from doubly-ionized atoms of Cd, Cu, Zn, Pb and Ca, from salts in which these metals appear as divalent.—[T. Volmer, Münster; ZS. f. Phys. *26*, pp. 285-288; 1924.]

KARL K. DARROW

[6] Arnold and Elmen, Jour. Frank. Inst., *195*; May, 1923.
[7] Ross, Trans. Faraday Soc., *8*, p. 185; Oct. 1912; *19*, p. 1; 1913.
[8] Young, Phil. Mag., p. 291; Aug. 1923.
[9] Hull, Phys. Rev., *14*, p. 540; 1919.

METHODS FOR MEASURING HUMIDITY*

By S. P. Fergusson

Humidity may be measured or determined (1) by the chemical or "absolute" method, whereby the actual amount of water in a given space, or its pressure, is obtained by means of absorptive chemicals, (2) from the temperature of condensation, (3) by the psychrometric method (dry-and-wet-bulb thermometers) or (4) by the change in volume of hygroscopic materials (the hair-hygrometer). Values of the maximum vapor-pressure (which depends entirely upon the temperature and upon which are based all standard formulæ and tables used in hygrometry) have been determined, chiefly by the chemical method, by Regnault (Annales de Chimie,) (series *3*,) *14*, 1845; W. N. Shaw (Phil. Trans. Roy. Soc., *179*, p. 73; 1888), Marvin (Report, Chief Signal Officer U. S. A., 1891), and recently by Scheele and Heuse (Ann. der Physik, *26*, *29* and *31*; 1908 to 1910), whose results, obtained at temperatures between $-70°$ and 374°C are probably the most accurate available. A very thorough investigation, chiefly of the psychrometric method, for industrial use, at temperatures above the atmospheric range, has been conducted by D. C. Lindsay at Cornell University, but no results have been published.

(1) The Chemical Method. The chemical method has been generally employed in standardizing other methods. By means of an aspirator, a measured quantity of air is drawn through tubes containing drying materials (recently-fused calcium chloride or pumice saturated with strong sulphuric acid); the water absorbed from the air is measured by weighing the tubes (the "gravimetric" process), or its vapor-pressure determined by means of a sensitive gauge (the "volumetric" method). An improved apparatus compensated for small variations of pressure and temperature, has been devised by A. Norman Shaw (Trans. Roy. Soc. Canada, *10*; 1916), who comments as follows upon the accuracy of the chemical method: "If the humidity remains constant it can be determined . . . to within 0.1 per cent of its true value. This requires, however, expert attention and "research" conditions. Under ordinary conditions it is difficult to eliminate errors of less than two or three per cent. These methods are almost useless for the examination of a

* One of a series of reports prepared at the request of the Committee on Research Methods and Technique appointed by the Division of Physical Sciences of the National Research Council. It is the purpose of these reports to present in brief outline a summary, more or less critical, of the methods which may be employed in various kinds of Physical measurements. See Editorial in this Journal, *9*, p. 410, 1924.

rapidly varying humidity, giving merely the average value during the time of observation."

(2) THE CONDENSATION METHOD. Alluard's modification of Regnault's dew-point hygrometer is probably the best-known apparatus. A silver tube, having a highly-polished flat face surrounded by a polished plate in the same plane as the face but not touching it, is filled with ether which is cooled by bubbling air through it until condensation of moisture is indicated by a change in the appearance of the face, the surrounding plate remaining unaltered; a thermometer immersed in the ether is read at the instant condensation appears. The tube is then allowed to warm and the thermometer is read at the instant the moisture disappears; the mean of the two readings is assumed to be the dew-point. (Jour. de Physique, 1878.) Improved instruments by Crova and others have devices for eliminating effects of the wind, etc.

The accuracy of this method depends (1) upon the skill of the observer in detecting the precise instant of appearance or disappearance of condensation, (2) the accuracy of the thermometer and (3) the velocity of the wind; also, to avoid the very marked effect of his body upon the moisture and temperature of the air surrounding the apparatus, the observer, according to Ezer Griffiths, should make all readings of thermometers, etc. through a telescope from a distance of ten meters. "Under the most favorable circumstances an accuracy of greater than 1 per cent can not be assured. If the relative humidity is less than 20 per cent or if the temperature is below 5°C errors as large as 10 per cent may easily occur."—A. Norman Shaw.

(3) THE PSYCHROMETRIC METHOD. For the best results the errors of the thermometers must be determined and readings to tenths of degrees should be possible; the covering of the wet-bulb should be of thin, closely-woven linen or muslin from which all grease or dirt has been removed by boiling and rinsing. The psychrometer should be exposed in air moving at least 3 meters a second; proper ventilation is most easily obtained by using a sling or whirling psychrometer or an aspiration instrument of the Assmann type.

This method is most accurate at atmospheric temperatures above 0°C but may be employed at much lower temperatures; the wet-bulb then will indicate the temperature of evaporating ice. "Under the best conditions a good constancy of repetition may be obtained but an absolute accuracy of within 2 per cent cannot be assured. If care of the wet sack and attention to its ventilation are neglected, errors of greater than 5 per cent may develop under apparently normal conditions. If

either the relative humidity or the temperature is low, or if the relative humidity is approaching 100 per cent the errors may be larger than 10 per cent."—A. Norman Shaw. Uncertainties, due to errors of observation and the smallness of the quantities to be measured, are so large at very low temperatures that other methods often are preferable. At $-15°C$, for example, a difference between the dry and wet thermometers of $0.1°$ is equal to about 5 per cent relative humidity.

(4) THE HAIR-HYGROMETER. Human hair, preferably of small size and light color (blonde), has been found to be the most satisfactory material for indicating and recording relative humidity, particularly at low temperatures where the psychrometric and other methods fail. After it is sensitized by soaking in ether about 24 hours or less time, hair will change in length about 2.5 per cent when exposed alternately to saturated and very dry air, and will be more sensitive to rapid variations of humidity than a psychrometer; it is extremely durable, sometimes retaining its sensitiveness many years, and practically independent of changes of temperature; it is elastic, consequently, in both indicating and recording instruments the tension of the hairs and friction of pivots must be constant in order to secure uniform scale-values. Practically the only serious defect of hair as a hygrometric material is its variable zero which necessitates occasional comparisons with a psychrometer or other instrument with a fixed zero.

The simplest instruments, such as Koppe's, are often the most satisfactory, since a single hair operating a light index is easily adjusted or replaced when damaged; but a number of hairs or a bundle will be necessary to actuate the pen of a hygrograph.

TABLES, in English and International measures, for use with the methods described, are published in this country by the United States Weather Bureau and the Smithsonian Institution, and in other countries by various meteorological organizations.

BIBLIOGRAPHY. The papers already referred to, the excellent "Discussion on Hygrometry," by Napier Shaw, Simpson, Griffiths, and others (Proc. Phys. Soc. London, 1922), the article on "Humidity" by Skinner, in "A Dictionary of Applied Physics," *3*, (Glazebrook and others, London, 1923), and other important memoirs freely drawn upon for the material in this paper, contain valuable information concerning the methods and applications of hygrometry as well as extensive bibliographies.

UNITED STATES WEATHER BUREAU,
WASHINGTON, D. C.
FEBRUARY 8, 1924.

Helmholtz's Treatise on Physiological Optics.—Translated from the Third German Edition. Edited by James P. C. Southall, Professor of Physics in Columbia University. Volume I. With Portrait of Helmholtz. XXI+482 pages. Published by The Optical Society of America. 1924.

"Most of those who have risen to eminence in any one of the sciences have done so by devoting their whole attention to that science exclusively, so that it is only rarely that the cultivators of different branches can be of service to each other by contributing to one science the skill they have acquired by the study of another.

"Hence the ordinary growth of human knowledge is by accumulation round a number of distinct centres. The time, however, must sooner or later arrive when two or more departments of knowledge can no longer remain independent of each other, but must be fused into a consistent whole. But though men of science may be profoundly convinced of the necessity of such a fusion, the operation itself is a most arduous one. For though the phenomena of nature are all consistent with each other, we have to deal not only with these, but with the hypotheses which have been invented to systematise them; and it by no means follows that because one set of observers have laboured with all sincerity to reduce to order one group of phenomena, the hypotheses which they have formed will all be consistent with those by which a second set of observers have explained a different set of phenomena. Each science may appear tolerably consistent within itself, but before they can be combined into one, each must be stripped of the daubing of untempered mortar by which its parts have been prematurely made to cohere.

"Hence the operation of fusing two sciences into one generally involves much criticism of established methods, and the explosion of many pieces of fancied knowledge which may have been long held in scientific reputation."

With these general observations Professor James Clerk Maxwell prefaced his appreciation of Helmholtz, (Nature, *15*), and, having in mind the greatest classical illustrations of these principles, the "Physiological Optics" and the "Sensations of Tone," applied them to the great personality he was contemplating by remarking that Helmholtz "is himself the most illustrious example not merely of extensive acquaintance with science combined with thoroughness, but of a thoroughness which of itself demands the mastery of many sciences, and in so doing makes its mark on each."

Since the time of both Helmholtz and Maxwell science has not been lacking in great investigators in all its branches. But such men as Helmholtz rarely appear whose intellectual powers can explore with equal facility such diversified regions of learning as Mathematics, Physics, Physiology, Psychology, and Aesthetics, contribute profoundly to the extension of each in its own field, and combine them into one

comprehensive whole for the elucidation of some great fundamental problem such as, in the present instance, vision.

It was in the year 1866 that the complete edition of the *Handbuch der Physiologischen Optik*, begun in 1857, was given to the world. Since that time a second edition, in 1885, was issued by Helmholtz himself, in which he greatly extended his treatment of parts of his wide subject, and a posthumous third edition was brought out in 1909–11 under the capable editorial supervision of Professors Gullstrand, von Kries, and Nagel. In the last edition the text of the first was reproduced, and each of the editors contributed extensive sections dealing with those parts of visual science in which notable advances had been made since the time of Helmholtz. For reasons which are discussed in the Preface to the third edition, and translated in the English edition, the editors decided upon what on first consideration would appear to be the wrong course in ignoring the second edition of Helmholtz and choosing the original text. After reading the considerations that brought them to this decision, one is compelled to admit the wisdom of their choice. Even more justified was the publication committee of the Optical Society of America in following their example, since advantage could be taken of the extensive supplements by the editors of the German edition.

It must have been a cause of wonder to every English speaking investigator in visual science that in the long period of nearly sixty years no English translation of this monumental work had been undertaken, especially since an English translation of the companion treatise, the "Sensations of Tone," had been made by Ellis in 1875, only thirteen years after its first publication by Helmholtz.

There can be no doubt but that in Physiological Acoustics the ideas and theories of Helmholtz have dominated and directed scientific thought, not by inducing slavish deference to great authority, but by reason of the weight of the argument in which Helmholtz had consolidated all the evidence, both theoretical and experimental, that pertained to the subject, from all the contributory sciences in which he was so deeply versed.

While the *Handbuch der Physiologischen Optik* has undoubtedly been the most powerful stimulus in the modern development of visual science in all its manifold ramifications, yet it is equally true that its development has suffered among British and American investigators because by far the greatest treatise on the subject was not available in English, and therefore for the most part was only consulted and not read. So much, indeed, has this been the case, negativing even the consultation, that Professor Troland has remarked that "nearly every year some classical content of Helmholtz's Physiological Optics is rediscovered and published," a redundancy which he rightly says will be inexcusable with the issue of this translation.

At its Annual Meeting in 1921 the extremely youthful Optical Society of America decided to commemorate the centenary of the birth of

Helmholtz by the preparation and publication of an English translation of the Physiological Optics. Under the capable editorial management of Professor J. P. C. Southall of Columbia University, an eminent authority in Geometrical Optics, and with the cooperation of thirteen associates in the work of translation, whose names are recorded in the first volume of the work, the laborious undertaking has been accomplished with such rapidity that the first volume is now available, and the two remaining volumes are promised within a very short time. Such an achievement is a tribute to the energy and organizing ability of the editor, especially, when it appears, upon examination of the first volume, which is the subject of this review, that the quality of the translation and publication has not suffered in any way. In no part does it read like a mere translation from another language, but it impresses one as a work of independent literary composition. The reviewer cannot pretend to have compared throughout in so short a time the two texts of so voluminous a work; but in many scattered comparisons no inaccuracy was apparent. The reviewer has compared, he believes, every mathematical equation and numerical constant, except in the last section, with the German edition, and has noted only a few minor misprints. These appear at the end of this review. A few unimportant verbal misprints were also noticed and communicated directly to the editor.

The English translation therefore may be relied upon to the same extent as the German text in its exposition of the principles of the dioptrics of the eye, the mathematical equations, and the numerous important numerical constants of the eye.

In several places constants of the German edition have been corrected in the translation. In the original work measurements are given in Paris lines and inches, in Prussian and English inches, and in various denominations of the metric system. By preserving these no doubt comparison with the original sources from which they were taken is thereby facilitated. Though in many cases the metric equivalents have been inserted, one may raise the question whether fidelity to the German text might not have been sacrificed to the extent of uniformly restating all measurements in the now customary scientific units.

Professor Southall has inserted throughout the volume a number of valuable footnotes to correct or to explain ambiguities in the text. In one of these, however, he gives the limiting visible wave length in the red as 723mμ, (p. 50). While the exact limit is somewhat uncertain on account of variations in the light and in the eye, yet the normal observer can frequently see the line A which has a wave length of 759mμ. Other observers give the limit somewhere from 800mμ to 830mμ, and Dr. Nutting states that under the most favorable conditions of intensity of the light and of the adaptation of the eye, the outer limit may be extended to 900mμ.

In dealing with an intricate subject like the dioptrics of the eye there is a possibility that confusion may arise from the expressions used in

reference to the various optical measurements involved. Professor Southall accordingly has exercised discriminating care in the translation of such terms as Brennpunkten, Hauptpunkten, Hauptbrennpunkten, Fokalpunkte, and others of a similar nature, and in a note (p. 266) directs particular attention to the necessity of distinguishing carefully between them.

One advance in our understanding of the eye which is briefly discussed by Professor Gullstrand may be worthy of quotation from its historical interest, and from the fact that it affords an interesting illustration of the way in which the viewpoint in science may radically change with increasing knowledge. This refers to Helmholtz's criticism, and condemnation, as usually interpreted, of the eye as a defective optical instrument, which was given in his lecture on "Recent Progress of the Theory of Vision." Professor Gullstrand recalls this in his supplement: "Helmholtz's famous dictum, that the monochromatic aberrations of the eye are such as would not be tolerated in any good optical instrument, is sometimes construed to mean that the eye is a very badly constructed optical affair—which Helmholtz never said and certainly did not mean. But another question that this statement raises is whether these aberrations are not serviceable and what is their purpose" (p. 440). With the very modern development of the "laws of optical imagery in media of variable index of refraction," such as characterise the eye, these defects assume an entirely new aspect. Continuing the quotation, Professor Gullstrand adds: "Thus, we see again that the complicated aberrations of higher order that are present in the eye and the astonishingly large positive aberration within the optical zone do not impair the visual acuity with a pupil of the size mentioned. However, the dioptrics of the crystalline lens tells us that the former aberration is due entirely, and the latter in large measure, to the fact that the lens is composed of a heterogenous medium. The great advantage of such a medium is the elevation of the total index that takes place in accommodation, which denotes a change of optical focusing that is out of proportion to the change of form, and unattainable with homogeneous media *caeteris paribus*. The monochromatic aberrations are the necessary evil for obtaining this advantage; and even if the convergence of the rays is not so good as it might be, the clearness of the image in good illumination is still above the limit of the capacity of the eye as imposed by the laws of diffraction. Hence, the monochromatic aberrations are a witness for the perfection of the eye, if what is meant by perfection of an optical instrument is good convergence of rays to the degree that is needed to obtain the greatest useful sharpness of image; anything in excess of this being sacrificed in order to gain some other end" (p. 442).

Among the topics discussed in the supplementary sections is the theory of accommodation advanced by Tscherning to replace that of Helmholtz. Professor Gullstrand strongly criticises it, and states: "Modern investigations have established that the theory of mechanism of accommodation remains unchanged, in all essential features, just as

Helmholtz, by a real inspiration of genius, considering the state of knowledge of that time, conceived it" (p. 408).

Professor Southall has included in the supplementary part not only the original contributions of Professor Gullstrand to the German edition, but also an important section of forty pages by the same author further developing the extensive modern investigations on Ophthalmoscopy in which he is an outstanding authority. A new bibliography on the Anatomy of the eye, containing about two hundred references, was also prepared by Dr. Hooker, especially for this volume.

In consequence of these additions, and also because a somewhat larger and even more readable type was used than in the German edition, this volume has 482 pages of text, compared with 376 in its German prototype. There seems to be nothing to suggest in regard to the quality of the paper, the printing, or the binding, by which the publishers could have improved the appearance of the work.

This excellent English translation of Helmholtz's great treatise of necessity must find a place in every university and consulting library in the English-speaking world, and in the private libraries of those who are concerned with the theoretical development of the supremely important subject of the dioptrics of the eye. For those who are concerned with the application of this knowledge as practising Ophthalmologists and Optometrists, who desire to rise above the mere routine of their profession, the treatise is especially valuable, since not only is the history of the subject sufficiently outlined, but the conclusions of Helmholtz, as well as the modern ideas, are presented in as direct a form as possible without intricate mathematical analysis.

MISPRINTS

Page 181, Equation 1 (4 c), one symbol is broken.
Page 182, Equation 1(4*d*), n_3 occurs instead of n_4.
Page 243, Last Equation, x is omitted.
Page 372, Last equation, A is used instead of German letter.
Page 10, In last column of table, second value 3.9; should be 2.9;.
Page 152, In second column of table, fourth value 5.6 should be 3.2.
Page 220, Line 29. 0.32 should be 0.36.
Page 336, In fourth column of table, 3.662 should be 3.622.
Page 353, Line 29. 74.88 should be 74.58.

FRANK ALLEN